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Construction of Modular Branching Functions from Bethe's Equations in the 3-State Potts Chain

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We use the single-particle excitation energies and the completeness rules of the 3-state antiferromagnetic Potts chain, which have been obtained from Bethe's equation, to compute the modular invariant partition function. This provides a fermionic construction for the branching functions of the D_4 representation of Z_4 parafermions which complements the bosonic constructions. It is found that there are oscillations in some of the correlations and a new connection with the field theory of the Lee–Yang edge is presented.

KEY WORDS: Affine Lie algebras; conformal field theory; parafermions; modular invariant partition function; quasiparticles.

1. INTRODUCTION

The theory of integrable quantum spin chains was initiated by Bethe in 1931[1] in his study of the spin-1/2 Heisenberg antiferromagnet. From that beginning, particularly in the last 20 years, an enormous number of one-dimensional quantum spin systems have been discovered which, along with their two-dimensional statistical counterparts, have the remarkable property that their energy eigenvalues are given by the solutions of a system of equations which have become known as Bethe's equations:

$$(-1)^{M+1}\left[\frac{\sinh(\lambda_j - iS\gamma)}{\sinh(\lambda_j + iS\gamma)}\right]^N = \prod_{k=1}^{L} \frac{\sinh(\lambda_j - \lambda_k - i\gamma)}{\sinh(\lambda_j - \lambda_k + i\gamma)} \tag{1.1}$$

Here M is the number of sites in the chain, N and L are related to M (-typically $N = M$ or $2M$), and S and γ are parameters which characterize the specific models.

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One of the features derived from the Bethe equations is that in the limit $M \to \infty$ the spectrum of low-lying excitations E_{ex} above the ground state is expressed in terms of a set of single-particle levels $e_\alpha(P)$ depending on a momentum P and combined with a set of rules as

$$E_{ex} - E_{GS} = \sum_{\alpha, \text{rules}} e_\alpha(P_i) \tag{1.2}$$

$$P = \sum_{\alpha, \text{rules}} P_i^\alpha \tag{1.3}$$

and almost without exception one of the rules of combination is a "Fermi" exclusion rule:

$$P_i^\alpha \neq P_j^\alpha \qquad \text{if} \quad i \neq j \tag{1.4}$$

The form for the energy levels (1.2) and (1.3) is referred to as a quasiparticle spectrum. Furthermore, in many of these spin chains one or more e_α vanish as $P \to 0$,

$$e(P) \sim v\,|P| \tag{1.5}$$

where v is positive and is called the speed of sound or the Fermi velocity.

Much more recently, in 1984, a powerful new formalism was invented[(2)] to study those integrable systems for which there is no mass gap and (1.5) holds. This method, known as conformal field theory, is more axiomatic. It deals with a continuum approximation to the spin chain (or two-dimensional statistical system) and instead of starting from a Hamiltonian it starts from a symmetry principle such as the Virasoro algebra or Kac–Moody algebra, supplemented by modular invariance, which captures many of the essential features associated with the integrability of the systems described by Bethe's equation (1.1).

One of the main objects of computation in conformal field theory is the partition function, which is expressed[(3,4)] in terms of Virasoro characters or (more generally) branching functions $b_j(q)$ as

$$Z = \sum I_{k,l} b_k(q)\, b_l(\bar{q}) \tag{1.6}$$

where $I_{k,l}$ are nonnegative integers. Here q ($\bar{q}$) refers to the right (left) moving excitations with

$$q, \bar{q} = e^{-2\pi v / M k_B T} \tag{1.7}$$

where T is the temperature and k_B is Boltzmann's constant.

The Virasoro characters and branching functions $b_i(q)$ are solutions to the equations of modular transformation.[(5–8)] Their construction typically starts with one or more free boson Fock spaces, and then excludes certain null vectors. They are typically given by explicit formulas with several powers of the product

$$Q(q) = \prod_{n=1}^{\infty} (1 - q^n) \tag{1.8}$$

in the denominator, and a power series in q in the numerator, times a fractional power which is usually written as $q^{-c/24 + h_k}$. Here the constant c is referred to as the central charge, and h_k are known as the conformal dimensions.

The question now arises as to the relation between the solutions of Bethe's equations and the results of conformal field theory. In particular, one wants to compute the partition function (1.6) starting from Bethe's equations (1.1) (or related functional equations). Recently an important advance in this project was made by Klümper and Pearce,[9,10] who computed the central charge and conformal dimensions for the A_{N+1} series of the $A_1^{(1)}$ models classified by Pasquier.[11] However, the computation of the full character expansion (1.6) and clarification of its relation to the quasi particle energy spectrum (1.2) is still lacking.

It our purpose here to complete this project and to compute the full partition function (1.6) for a particular quantum spin model: the antiferromagnetic 3-state Potts chain. In particular, we will show that the partition function is constructed from the single-particle levels of (1.2) which satisfy the Fermi exclusion rule (1.4). This provides a physical interpretation of the model, which complements the usual computation that starts with free bosons. The result of conformal field theory, as obtained by Pearce,[12] is that the partition function of the spin system is

$$Z = \sum e^{-E_n/k_B T} = e^{-Me_0/k_B T} Z_{pf4} \tag{1.9}$$

where e_0 is the ground-state energy per site[13] and Z_{pf4} is the D_4 representation of the Z_4 parafermionic partition function of ref. 14:

$$\begin{aligned} Z_{pf4} = {} & [b_0^0(q) + b_4^0(q)][b_0^0(\bar{q}) + b_4^0(\bar{q})] \\ & + 4b_2^0(q)\, b_2^0(\bar{q}) + 2b_0^2(q)\, b_0^2(\bar{q}) + 2b_2^2(q)\, b_2^2(\bar{q}) \end{aligned} \tag{1.10}$$

with b_m^l given in one of the equivalent bosonic forms of (2.25), (2.27), and Appendix B. We here obtain (1.10) starting from Bethe's equation (1.1) for the finite lattice and obtain fermionic representations for b_m^l given by (3.13) and (3.19) for b_0^0, b_2^0, and b_4^0 in $Q = 0$ and (4.15) for b_0^2 and b_2^2 in $Q = \pm 1$. In $Q = 0$ the form (3.13) agrees with the form of the branching functions of $A_1^{(1)}$ given by Lepowsky and Primc.[15]

Our method is to combine the results of ref. 13, which derives the spectrum of the antiferromagnetic 3-state Potts model of the form (1.2), starting from the Bethe equation derived by Albertini,[16] with the completeness study of ref. 17 and the finite-size corrections of ref. 10. In Section 2 we summarize the results of these papers that are needed here, as well as the conformal field theory predictions for the model. In Section 3 we compute the partition function in the channel $Q=0$ and in Section 4 we do the same in the channel $Q=1$. We conclude in Section 5 with a discussion of the physical implications of our results. We also discuss the oscillations which are predicted to occur in the correlation functions, and a connection with the field theory[18,19] of the Lee–Yang edge[20] of the Ising model.

2. FORMULATION

The 3-state antiferromagnetic Potts chain is specified by the Hamiltonian

$$H=\frac{2}{\sqrt{3}}\sum_{j=1}^{M}\{X_j+X_j^\dagger+Z_jZ_{j+1}^\dagger+Z_j^\dagger Z_{j+1}\} \tag{2.1}$$

where

$$X_j=I\otimes I\otimes\cdots\otimes \underbrace{X\otimes}_{j\text{th}}\cdots\otimes I \tag{2.2}$$

$$Z_j=I\otimes I\otimes\cdots\otimes \underbrace{Z\otimes}_{j\text{th}}\cdots\otimes I \tag{2.3}$$

Here I is the 3×3 identity matrix, the elements of the 3×3 matrices X and Z are

$$X_{j,k}=\delta_{j,k+1} \pmod 3 \tag{2.4}$$

$$Z_{j,k}=\delta_{j,k}\omega^{j-1} \tag{2.5}$$

$$\omega=e^{2\pi i/3} \tag{2.6}$$

and we impose periodic boundary conditions $Z_{M+1}\equiv Z_1$.

This spin chain is invariant under translations and under spin rotations. Thus the eigenvalues may be classified in terms of P, the total momentum of the state, and Q, where $e^{2\pi iQ/3}$ is the eigenvalue of the spin rotation operator. Here $P=2\pi n/M$, where n is an integer $0\leqslant n\leqslant M-1$, and $Q=0, \pm 1$. Furthermore, because H is invariant under complex conjugation there is a conserved C parity of ± 1 in the sector $Q=0$ and the sectors $Q=\pm 1$ are degenerate.

This spin chain is integrable because of its connection with the integrable 3-state Potts model of statistical mechanics. The eigenvalues satisfy functional equations[16,21–23] which are solved in terms of a Bethe equation (1.1)[16] with

$$N=2M, \qquad \gamma=\pi/3, \qquad S=1/4 \tag{2.7}$$

and

$$L=2(M-|Q|) \qquad \text{for} \quad Q=0, \pm 1 \tag{2.8}$$

In terms of these λ_k, the eigenvalues of the transfer matrix of the statistical model are

$$\Lambda(\lambda)=\left[\frac{\sinh(\pi i/6)\sinh(\pi i/3)}{\sinh(\pi i/4-\lambda)\sinh(\pi i/4+\lambda)}\right]^M \prod_{k=1}^{L} \frac{\sinh(\lambda-\lambda_k)}{\sinh(\pi i/12+\lambda_k)} \tag{2.9}$$

the eigenvalues of the Hamiltonian (2.1) are

$$E=\sum_{k=1}^{L} \cot\left(i\lambda_k+\frac{\pi}{12}\right)-\frac{2M}{\sqrt{3}} \tag{2.10}$$

and the corresponding momentum is

$$e^{iP}=\Lambda\left(\frac{-i\pi}{12}\right)=\prod_{k=1}^{L}\frac{\sinh(\lambda_k+\pi i/12)}{\sinh(\lambda_k-\pi i/12)} \tag{2.11}$$

These equations have been solved to find the order-one excitation energies.[13] The results are expressed in terms of three single-particle excitation energies:

$$e_{2s}(P)=3\left\{\sqrt{2}\cos\left(\frac{|P|}{2}-\frac{3\pi}{4}\right)+1\right\} \tag{2.12a}$$

$$e_{-2s}(P)=3\left\{\sqrt{2}\cos\left(\frac{|P|}{2}-\frac{\pi}{4}\right)-1\right\} \tag{2.12b}$$

$$e_{ns}(P)=3\sin\left(\frac{|P|}{2}\right) \tag{2.12c}$$

For $P \sim 0$, all three excitations are of the form (1.5) with

$$v=3/2 \tag{2.13}$$

Here and in the remainder of the paper we take M to be even.

1. For $Q=0$ the energies and momenta are of the form (1.2) and (1.3),

$$E(\{P_j^{2s}\}, \{P_j^{-2s}\}, \{P_j^{ns}\}) - E_{\mathrm{GS}} = \sum_{\alpha=2s,\,-2s,\,ns} \sum_{j=1}^{m_\alpha} e_\alpha(P_j) \tag{2.14}$$

and

$$P = P^0 + \sum_{\alpha=2s,\,-2s,\,ns} \sum_{j=1}^{m_\alpha} P_j^\alpha \tag{2.15}$$

where

$$P^0 = P_{\mathrm{GS}} = \frac{M}{2}\pi \mod 2\pi \tag{2.16}$$

$$m_{2s} + m_{-2s} \text{ is even} \tag{2.17}$$

P_j^{2s}, P_j^{-2s}, and P_j^{ns} obey the Fermi exclusion rule (1.4) and they lie in the ranges

$$0 \leqslant P_j^{2s} \leqslant 3\pi \tag{2.18a}$$

$$0 \leqslant P_j^{-2s} \leqslant \pi \tag{2.18b}$$

$$0 \leqslant P_j^{ns} \leqslant 2\pi \tag{2.18c}$$

We also note that the C parity of the ground state is

$$C_{\mathrm{GS}} = (-1)^{M/2} \tag{2.19}$$

and the C parity of an arbitrary state is

$$C/C_{\mathrm{GS}} = (-1)^{m_{ns} + m_{-2s} + (m_{2s} + m_{-2s})/2} \tag{2.20}$$

2. For $Q = \pm 1$ we must consider $m_{2s} + m_{-2s}$ to be both even and odd. When $m_{2s} + m_{-2s}$ is even there are two spectra of the form (2.14) and (2.15). In one $P^0 = P_{\mathrm{GS}}$ and in the other $P^0 = P_{\mathrm{GS}} + \pi$. In both cases the P_j^α obey (2.18). When $m_{2s} + m_{-2s}$ is odd, there are again two spectra of the form (2.14) and (2.15). In each case $P^0 = P_{\mathrm{GS}}$. In one case P_j^α satisfies (2.18), while in the other case $-P_j^\alpha$ satisfies (2.18).

The conformal field theory predictions for the model can be obtained by noting that the 3-state Potts model is the critical D_4 model in the classification of Pasquier.[11] The central charge and the conformal dimensions of the primary fields are thus obtained by specializing the finite-size computations of the A_{N+1} model of Klümper and Pearce[10] to the case $N=4$ and using an orbifold construction[24,25] to find the results for D_4. The general result for A_{N+1} at the boundary of the I/II regime is that the central charge is

$$c=\frac{2(N-1)}{N+2} \tag{2.21}$$

and the conformal dimensions are

$$h_m^l=\frac{l(l+2)}{4(N+2)}-\frac{m^2}{4N} \qquad \text{for} \quad |m|\leqslant l \tag{2.22}$$

which are the same as those of the Z_N parafermion conformal field theory.[26,27] Using the symmetry $h_m^l=h_{N-m}^{N-l}$, we find for $N=4$

$$c=1 \tag{2.23}$$

and

$$h_0^0=0, \qquad h_2^0=\tfrac{3}{4}, \qquad h_4^0=1, \qquad h_0^2=\tfrac{1}{3}, \qquad h_2^2=\tfrac{1}{12} \tag{2.24}$$

where the first three conformal dimensions occur in $Q=0$ and the last two in $Q=\pm 1$. Moreover, the modular invariant partition function is that of the D_4 parafermion model,[14] (1.10) where the branching functions b_m^l can be obtained by specializing to $N=4$ the Hecke indefinite form of Kac and Peterson[7] of $(A_1^{(1)})_N/U(1)$,

$$\begin{aligned} b_m^l=\,&Q(q)^{-2}q^{l(l+2)/4(N+2)-m^2/4N-c/24}\\ &\times\Bigg[\Bigg(\sum_{s\geqslant 0}\sum_{n\geqslant 0}-\sum_{s<0}\sum_{n<0}\Bigg)(-1)^s q^{s(s+1)/2+(l+1)n+(l+m)s/2+(N+2)(n+s)n}\\ &+\Bigg(\sum_{s>0}\sum_{n\geqslant 0}-\sum_{s\leqslant 0}\sum_{n<0}\Bigg)(-1)^s q^{s(s+1)/2+(l+1)n+(l-m)s/2+(N+2)(n+s)n}\Bigg] \end{aligned} \tag{2.25}$$

for $|m|\leqslant l$, and using the symmetries

$$b_m^l=b_{-m}^l=b_{m+2N}^l=b_{N-m}^{N-l} \tag{2.26}$$

otherwise. An alternative form for b_m^l is given in ref. 28, but for our purposes the simplest form is the specialization which only occurs for $N=4$,[8]

$$\begin{gathered} b_0^0+b_4^0=f_{3,0}/\eta, \qquad b_0^0-b_4^0=g_{1,0}/\eta \\ b_2^0=f_{3,3}/2\eta, \qquad b_0^2=f_{3;2}/\eta, \qquad b_2^2=f_{3,1}/\eta \end{gathered} \tag{2.27}$$

where

$$\eta = q^{1/24} Q(q) \tag{2.28}$$

and

$$f_{a,b} = \sum_{n=-\infty}^{\infty} q^{a(n+b/2a)^2}, \qquad g_{a,b} = \sum_{n=-\infty}^{\infty} (-1)^n q^{a(n+b/2a)^2} \tag{2.29}$$

This form has a simple origin in the Gaussian model with $r=(3/2)^{1/2}$, which we give in Appendix A. For comparison with the expansions of subsequent sections, we list the first terms of (2.27) as

$$\begin{aligned} q^{1/24} b_0^0 = {} & (1+q^2+2q^3+4q^4+5q^5+9q^6 \\ & +12q^7+19q^8+25q^9+37q^{10}+\cdots) \end{aligned} \tag{2.30a}$$

$$\begin{aligned} q^{1/24} b_2^0 = {} & q^{3/4}(1+q+2q^2+3q^3+5q^4+7q^5+12q^6 \\ & +16q^7+24q^8+33q^9+47q^{10}+\cdots) \end{aligned} \tag{2.30b}$$

$$\begin{aligned} q^{1/24} b_4^0 = {} & q(1+q+3q^2+3q^3+6q^4+8q^5+13q^6 \\ & +17q^7+27q^8+35q^9+51q^{10}+\cdots) \end{aligned} \tag{2.30c}$$

$$\begin{aligned} q^{1/24} b_0^2 = {} & q^{1/3}(1+2q+3q^2+5q^3+8q^4+13q^4+19q^6 \\ & +28q^7+41q^8+58q^9+81q^{10}+\cdots) \end{aligned} \tag{2.30d}$$

$$\begin{aligned} q^{1/24} b_2^2 = {} & q^{1/2}(1+q+3q^2+4q^3+8q^4+11q^5+18q^6 \\ & +25q^7+38q^8+52q^9+76q^{10}+\cdots) \end{aligned} \tag{2.30e}$$

We finally note that there is an alternative way to obtain these conformal field theory predictions which utilizes W_4 algebra[29] and is related to the GKO construction $(A_3^{(1)})_1 \times (A_3^{(1)})_1/(A_3^{(1)})_2$. The branching functions have been computed from this construction in terms of three-dimensional sums[30–32] which, for later comparison, we give in Appendix B. The equality of these branching functions with those of $(A_1^{(1)})_4/U(1)$ is a consequence of level rank duality.[33]

3. BRANCHING FUNCTIONS FOR $Q=0$

The partition function for the Hamiltonian (2.1) is, by definition,

$$Z = \sum_n e^{-E_n/k_B T} = e^{-E_{GS}/k_B T} \sum e^{-(E_n - E_{GS})/k_B T} \tag{3.1}$$

where, for $M \to \infty$,

$$E_{GS} = Me_0 - \frac{\pi c v}{6M} + O\left(\frac{1}{M^2}\right) \tag{3.2}$$

and from ref. 10, $c=1$. To obtain the relation with the modular invariant partition function of conformal field theory we must evaluate (3.1) in the limit $M \to \infty$, $T \to 0$ with MT fixed. We intend to carry out this evaluation by making use of the quasiparticle energy spectrum (2.14).

There are, however, two questions that must be addressed before we can do this. The first is that in order for (2.14) to specify the energy levels, the momenta P_j^α must be discretely specified on the finite lattice. The second is that the evaluation leading to (2.14) is only correct to order one as $M \to \infty$, and hence in order to agree with ref. 10, it may be necessary to add some term of the order of $1/M$ which is independent of P_j^α but will in general depend on m_{2s}, m_{-2s}, and m_{ns}. These considerations are different for $Q=0$ and $Q=\pm 1$.

We consider in this section $Q=0$. We find from the previous study[17] of the completeness of the solutions of (2.1) that, for given m_{2s}, m_{-2s}, and m_{ns},

$$P_j^{2s} \quad \text{takes} \quad \frac{3M}{2} - m_{ns} - \frac{m_{2s}+m_{-2s}}{2} \quad \text{values} \tag{3.3a}$$

$$P_j^{-2s} \quad \text{takes} \quad \frac{M}{2} - m_{ns} - \frac{m_{2s}+m_{-2s}}{2} \quad \text{values} \tag{3.3b}$$

and

$$P_j^{ns} \quad \text{takes} \quad M - m_{ns} - m_{2s} - m_{-2s} \quad \text{values} \tag{3.3c}$$

This will be the case if P_j^α satisfies

$$\frac{\pi}{M}\left(m_{ns} + \frac{m_{2s}+m_{-2s}}{2} + 1\right) \leqslant P_j^{2s} \leqslant 3\pi - \frac{\pi}{M}\left(m_{ns} + \frac{m_{2s}+m_{-2s}}{2} + 1\right) \tag{3.4a}$$

$$\frac{\pi}{M}\left(m_{ns} + \frac{m_{2s}+m_{-2s}}{2} + 1\right) \leqslant P_j^{-2s} \leqslant \pi - \frac{\pi}{M}\left(m_{ns} + \frac{m_{2s}+m_{-2s}}{2} + 1\right) \tag{3.4b}$$

$$\frac{\pi}{M}(m_{ns} + m_{2s} + m_{-2s} + 1) \leqslant P_j^{ns} \leqslant 2\pi - \frac{\pi}{M}(m_{ns} + m_{2s} + m_{-2s} + 1) \tag{3.4c}$$

where the spacing between allowed values for P_j^α is $2\pi/M$, the Fermi exclusion rule (1.4) holds, and $m_{2s}+m_{-2s}$ is even. It may be verified that this choice of P_j^α exactly reproduces the correct number of momenta of Table 4 of ref. 17 for each allowed set of m_{2s}, m_{-2s}, and m_{ns}.

Since $M \to \infty$ and $T \to 0$ with MT fixed, only those values of P_j^α where $e_\alpha(P)$ is small of the order $1/M$ contributes to (3.1). This occurs for

$$P_j^\alpha \sim 0 \qquad \text{for} \quad \alpha = 2s,\ -2s,\ ns \tag{3.5a}$$

$$P_j^{-2s} \sim \pi, \qquad P_j^{ns} \sim 2\pi, \qquad P_j^{2s} \sim 3\pi \tag{3.5b}$$

where we may linearize $e_\alpha\ (P)$ near the endpoints (3.5) as

$$e_\alpha(P) \sim vP^\alpha \qquad \text{for} \quad P^\alpha \text{ near zero} \tag{3.6a}$$

$$e_{-2s}(P) \sim v(\pi - P^{-2s}) \qquad \text{for} \quad P^{-2s} \text{ near } \pi \tag{3.6b}$$

$$e_{ns}(P) \sim v(2\pi - P^{ns}) \qquad \text{for} \quad P^{ns} \text{ near } 2\pi \tag{3.6c}$$

and

$$e_{2s}(P) \sim v(3\pi - P^{2s}) \qquad \text{for} \quad P^{2s} \text{ near } 3\pi \tag{3.6d}$$

Thus we let m^l_α be the number of P^α_j near zero and m^r_α be the number of P^α_j near the endpoints (3.5b). We note that

$$m^l_\alpha + m^r_\alpha = m_\alpha \tag{3.7}$$

We also note that if

$$m^r_{2s} + m^r_{-2s} \text{ is odd} \tag{3.8}$$

then from (2.15) the total momentum of the state is macroscopically shifted π from the ground-state value P_{GS}. These states are expected to make oscillatory contributions to the correlation functions.

Consider first the case where all $m^r_\alpha = 0$ (which by symmetry is identical to the case $m^l_\alpha = 0$) and evaluate the partition function (3.1) using (2.14), (3.4), and (3.6) in the case $C/C_{\text{GS}} = 1$, where, by (2.20),

$$m^l_{ns} + m^l_{-2s} + (m^l_{2s} + m^l_{/-2s})/2 \quad \text{is even} \tag{3.9}$$

We present in Table I the terms from this construction up through order q^8, where we see that they agree with the corresponding terms from the branching function $q^{1/24}b^0_0$ of (2.27). This equality has been verified to order q^{200} and thus we conclude that this construction correctly gives the branching function b^0_0.

To obtain a formula for b^0_0 from the construction, let $P_d(m, n)$ denote the number of distinct ways that the integer n can be additively partitioned into m distinct parts. Then, modifying the usual construction of a free Fermi partition function in terms of $P_d(m, n)$ to account for the momentum exclusion rule (3.4), we find

$$\begin{aligned} q^{1/24}b^0_0 = & \sum_{m_{ns}, m_{2s}, m_{-2s}=0}^{\infty} \ \sum_{n_{ns}, n_{2s}, n_{-2s}=0}^{\infty} P_d(m_{ns}, n_{ns})\, P_d(m_{2s}, n_{2s}) \\ & \times P_d(m_{-2s}, n_{-2s})\, q^{n_{ns}+n_{2s}+n_{-2s}} \\ & \times q^{(m_{ns}/2)(m_{ns}+m_{2s}+m_{-2s}-1)}\, q^{[(m_{2s}+m_{-2s})/2][m_{ns}+(m_{2s}+m_{-2s})/2-1]} \end{aligned} \tag{3.10}$$

where $m_{2s}+m_{-2s}$ and $m_{ns}+m_{-2s}+(m_{2s}+m_{-2s})/2$ are even and $P_d(0,0)=1$ by definition. The sums over n_α are evaluated using

$$\sum_{n=0}^{\infty} P_d(m,n)q^n = \frac{q^{m(m+1)/2}}{(q)_m} \tag{3.11}$$

Table I. The Terms through Order q^8 in the Construction of b_0^0 from the Rules of Section 3[a]

Order	m_{ns}^l	m_{2s}^l	m_{-2s}^l	$P_{\min}^{ns}$	$P_{\min}^{\pm 2s}$	$\{P^{ns}, P^{2s}, P^{-2s}\}$ (units of π/M)	States	$q^{1/24}b_0^0$
q^0	0	0	0	—	—	$\{0,0,0\}$	1	1
q^2	0	1	1	—	$2\pi/M$	$\{0,2,2\}$	1	1
q^3	0	1	1	—	$2\pi/M$	$\{0,4,2\}$, $\{0,2,4\}$	2	2
q^4	0	1	1	—	$2\pi/M$	$\{0,6,2\}$, $\{0,4,4\}$, $\{0,2,6\}$	3	
	2	0	0	$3\pi/M$	—	$\{3+5,0,0\}$	1	4
q^5	0	1	1	—	$2\pi/M$	$\{0,8,2\}$, $\{0,6,4\}$, $\{0,4,6\}$, $\{0,2,8\}$	4	
	2	0	0	$3\pi/M$	—	$\{3+7,0,0\}$	1	5
q^6	0	1	1	—	$2\pi/M$	$\{0,10,2\}$, $\{0,8,4\}$, $\{0,6,6\}$, $\{0,4,8\}$, $\{0,2,10\}$	5	
	2	0	0	$3\pi/M$	—	$\{3+9,0,0\}$, $\{5+7,0,0\}$	2	
	1	2	0	$4\pi/M$	$3\pi/M$	$\{4,3+5,0\}$	1	
	1	0	2	$4\pi/M$	$3\pi/M$	$\{4,0,3+5\}$	1	9
q^7	0	1	1	—	$2\pi/M$	$\{0,12,2\}$, $\{0,10,4\}$, $\{0,8.6\}$, $\{0,6,8\}$, $\{0,4,10\}$, $\{0,2,12\}$	6	
	2	0	0	$3\pi/M$	—	$\{3+11,0,0\}$, $\{5+9,0,0\}$	2	
	1	2	0	$4\pi/M$	$3\pi/M$	$\{6,3+5,0\}$, $\{4,3+7\}$	2	
	1	0	2	$4\pi/M$	$3\pi/M$	$\{6,0,3+5\}$, $\{4,0,3+7\}$	2	12
q^8	0	1	1	—	$2\pi/M$	$\{0,14,2\}$, $\{0,12,4\}$, $\{0,10,6\}$ $\{0,8,8\}$, $\{0,6,10\}$, $\{0,4,12\}$ $\{0,2,14\}$	7	
	2	0	0	$3\pi/M$	—	$\{3+13,0,0\}$, $\{5+11,0,0\}$, $\{7+9,0,0\}$	3	
	1	2	0	$4\pi/M$	$3\pi/M$	$\{8,3+5,0\}$, $\{6,3+7,0\}$, $\{4,3+9,0\}$, $\{4,5+7,0\}$	4	
	1	0	2	$4\pi/M$	$3\pi/M$	$\{8,0,3+5\}$, $\{6,0,3+7\}$ $\{4,0,3+9\}$, $\{4,0,5+7\}$	4	
	0	2	2	—	$3\pi/M$	$\{0,3+5,3+5\}$	1	19

[a] The minimum momenta $P_{\min}^{ns}=(\pi/M)(m_{ns}+m_{2s}+m_{-2s}+1)$ and $P_{ns}^{\pm 2s}=(\pi/M)\times(m_{ns}+(m_{2s}+m_{-2s})/2+1)$ are obtained from (3.15). The terms in $q^{1/24}b_0^0$ are obtained from (2.30a). Here $m_{2s}+m_{-2s}$ and $m_{ns}+m_{-2s}+(m_{2s}+m_{-2s})/2$ are even. The macroscopic momentum shift is $\Delta P=0$.

where we use the standard notation

$$(q)_m = \prod_{j=1}^{m} (1-q^j) \tag{3.12}$$

and $(q)_0 = 1$ by definition. Thus we find that when $m_{2s} + m_{-2s}$ is even and (3.9) holds

$$q^{1/24} b_0^0 = \sum_{m_{ns}=0} \sum_{m_{2s}=0} \sum_{m_{-2s}=0} \frac{q^{m_{ns}(m_{ns}+1)/2}}{(q)_{m_{ns}}} \frac{q^{m_{2s}(m_{2s}+1)/2}}{(q)_{m_{2s}}} \frac{q^{m_{-2s}(m_{-2s}+1)/2}}{(q)_{m_{-2s}}}$$
$$\times q^{(m_{ns}/2)(m_{ns}+m_{2s}+m_{-2s}-1)} q^{[(m_{2s}+m_{-2s})/2][m_{ns}+(m_{2s}+m_{-2s})/2-1]}$$
$$= \sum_{m_{ns}=0} \sum_{m_{2s}=0} \sum_{m_{-2s}=0} \frac{q^{(2m_{2s}^2+3m_{-2s}^2+4m_{ns}^2+4m_{ns}m_{2s}+4m_{ns}m_{-2s}+2m_{2s}m_{-2s})/4}}{(q)_{m_{ns}}(q)_{m_{2s}}(q)_{m_{-2s}}} \tag{3.13}$$

This form for the branching function b_0^0 is identical with the expression derived by Lepowsky and Primc in the context of characters of $A_1^{(1)}$.[15] Furthermore, the quadratic form in the exponent of (3.13) is obtained from the inverse Cartan matrix for the group A_3.

We may now extend these considerations to the general case where both some $m_\alpha^r \neq 0$ and some $m_\alpha^l \neq 0$. In this general case we note from the work of refs. 9 and 10 that the contributions to the energy from regions where (3.5a) holds and the region where (3.5b) holds are independent. Combining the above considerations, we have in general the expression for the low-lying energy levels in the $M \to \infty$ limit:

$$E_{\text{ex}} - E_{\text{GS}} = \sum_{\alpha=2s,-2s,ns} \left\{ \sum_{j=1}^{m_\alpha^l} e_\alpha(P_j^{l,\alpha}) + \sum_{j=1}^{m_\alpha^r} e_\alpha(P_j^{r,\alpha}) \right\} \tag{3.14}$$

where we define $P_j^{l,\alpha}$ and $P_j^{r,\alpha}$ to satisfy

$$\frac{\pi}{M}\left(m_{ns}^l + \frac{m_{2s}^l + m_{-2s}^l}{2} + 1\right) \leqslant P_j^{l,2s},\ P_j^{l,-2s} \tag{3.15a}$$

$$\frac{\pi}{M}(m_{ns}^l + m_{2s}^l + m_{-2s}^l + 1) \leqslant P_j^{l,ns} \tag{3.15b}$$

and

$$\frac{\pi}{M}\left(m_{ns}^r + \frac{m_{2s}^r + m_{-2s}^r}{2} + 1\right) \leqslant P_j^{r,2s},\ P_j^{r,-2s} \tag{3.16a}$$

$$\frac{\pi}{M}(m_{ns}^r + m_{2s}^r + m_{-2s}^r + 1) \leqslant P_j^{r,ns} \tag{3.16b}$$

where again the spacing between allowed values for $P_j^{l,r}$ is $2\pi/M$, the Fermi exclusion rule (1.4) holds, and $e_\alpha(P) = vP$ with v given by (2.13).

In Table II we consider the cases $m^r_{2s}=1$, $m^r_{-2s}=m^r_{ns}=0$ and $m^r_{-2s}=1$, $m^r_{2s}=m^r_{ns}=0$ and compute the contribution made to Z to order $q^{31/4}$ of the terms in (3.14) that involve only $P^{l,\alpha}_j$. From (2.17) we see that

$$m^l_{2s}+m^l_{-2s} \text{ is odd} \tag{3.17}$$

Table II. The Terms through Order $q^{31/4}$ in the Construction of b^0_2 from the Rules of Section 3[a]

Order	m^l_{ns}	m^l_{2s}	m^l_{-2s}	$P^{ns}_{\min}$	$P^{\pm 2s}_{\min}$	$\{P^{ns}, P^{2s}, P^{-2s}\}$ (units of π/M)	States	$q^{1/24}b^0_2$
$q^{3/4}$	0	0	1	—	$3\pi/2M$	{0, 0, 3/2}	1	1
$q^{7/4}$	0	0	1	—	$3\pi/2M$	{0, 0, 7/2}	1	1
$q^{11/4}$	0	0	1	—	$3\pi/2M$	{0, 0, 11/2}	1	
	1	0	1	$3\pi/M$	$5\pi/2M$	{3, 0, 5/2}	1	2
$q^{15/4}$	0	0	1	—	$3\pi/2M$	{0, 0, 15/2}	1	
	1	0	1	$3\pi/M$	$5\pi/2M$	{3, 0, 9/2}, {5, 0, 5/2}	2	3
$q^{19/4}$	0	0	1	—	$3\pi/2M$	{0, 0, 19/2}	1	
	1	0	1	$3\pi/M$	$5\pi/2M$	{3, 0, 13/2}, {5, 0, 9/2}, {7, 0, 5/2}	3	
	0	1	2	—	$5\pi/M$	{0, 5/2, 5/2+9/2}	1	5
$q^{23/4}$	0	0	1	—	$3\pi/2M$	{0, 0, 23/2}	1	
	1	0	1	$3\pi/M$	$5\pi/2M$	{3, 0, 17/2}, {5, 0, 13/2} {7, 0, 9/2}, {9, 0, 5/2}	4	
	0	1	2	—	$5\pi/2M$	{0, 9/2, 5/2+9/2}, {0, 5/2, 5/2+13/2}	2	7
$q^{27/4}$	0	0	1	—	$3\pi/2M$	{0, 0, 27/2}	1	
	1	0	1	$3\pi/M$	$5\pi/2M$	{3, 0, 21/2}, {5, 0, 17/2}, {7, 0, 13/2} {9, 0, 9/2}, {11, 0, 5/2}	5	
	0	1	2	—	$5\pi/2M$	{0, 13/2, 5/2+9/2}, {0, 9/2, 5/2+13/2} {0, 5/2, 5/2+17/2}, {0, 5/2, 9/2+13/2}	4	
	2	0	1	$4\pi/M$	$7\pi/2M$	{4+6, 0, 7/2}	1	
	0	0	3	—	$5\pi/2M$	{0, 0, 5/2+9/2+13/2}	1	12
$q^{31/4}$	0	0	1	—	$3\pi/2M$	{0, 0, 31/2}	1	
	1	0	1	$3\pi/M$	$5\pi/2M$	{3, 0, 25/2}, {5, 0, 21/2}, {7, 0, 17/2} {9, 0, 13/2}, {11, 0, 9/2}, {13, 0, 5/2}	6	
	0	1	2	—	$5\pi/2M$	{0, 5/2, 5/2+21/2}, {0, 5/2, 9/2+17/2} {0, 9/2, 5/2+17/2}, {0, 9/2, 9/2+13/2} {0, 13/2, 5/2+13/2}, {0, 17/2, 5/2+9/2}	6	
	2	0	1	$4\pi/M$	$7\pi/2M$	{4+8, 0, 7/2}, {4+6, 0, 11/2}	2	
	0	0	3	—	$5\pi/2M$	{0, 0, 5/2+9/2+17/2}	1	16

[a] The minimum momenta $P^{ns}_{\min}=(\pi/M)(m_{ns}+m_{2s}+m_{-2s}+1)$ and $P^{\pm 2s}_{\min}=(\pi/M)\times(m_{ns}+(m_{2s}+m_{-2s})/2+1)$ are obtained from (3.15). The terms in $q^{1/24}b^0_2$ are obtained from (2.30b). Here $m_{2s}+m_{-2s}$ is odd and $m_{ns}<m_{-2s}$. The macroscopic momentum shift is $\Delta P=\pi$.

Table III. The Terms through Order q^8 in the Construction of b_4^0 from the Rules of Section 3[a]

Order	m'_{ns}	m'_{2s}	m'_{-2s}	$P^{ns}_{\min}$	$P^{\pm 2s}_{\min}$	$\{P^{ns}, P^{2s}, P^{-2s}\}$ (units of π/M)	States	$q^{1/24}b_4^0$
q^1	1	0	0	$2\pi/M$	—	$\{2, 0, 0\}$	1	1
q^2	1	0	0	$2\pi/M$	—	$\{4, 0, 0\}$	1	1
q^3	1	0	0	$2\pi/M$	—	$\{6, 0, 0\}$	1	
	0	2	0	—	$2\pi/M$	$\{0, 2+4, 0\}$	1	
	0	0	2	—	$2\pi/M$	$\{0, 0, 2+4\}$	1	3
q^4	1	0	0	$2\pi/M$	—	$\{8, 0, 0\}$	1	
	0	2	0	—	$2\pi/M$	$\{0, 2+6, 0\}$	1	
	0	0	2	—	$2\pi/M$	$\{0, 0, 2+6\}$	1	3
q^5	1	0	0	$2\pi/M$	—	$\{10, 0, 0\}$	1	
	0	2	0	—	$2\pi/M$	$\{0, 2+8, 0\}$, $\{0, 4+6, 0\}$	2	
	0	0	2	—	$2\pi/M$	$\{0, 0, 2+8\}$, $\{0, 0, 4+6\}$	2	
	1	1	1	$4\pi/M$	$3\pi/M$	$\{4, 3, 3\}$	1	6
q^6	1	0	0	$2\pi/M$	—	$\{12, 0, 0\}$	1	
	0	2	0	—	$2\pi/M$	$\{0, 2+10, 0\}$, $\{0, 4+8, 0\}$	2	
	0	0	2	—	$2\pi/M$	$\{0, 0, 2+10\}$, $\{0, 0, 4+8\}$	2	
	1	1	1	$4\pi/M$	$3\pi/M$	$\{4, 5, 3\}$, $\{4, 3, 5\}$, $\{6, 3, 3\}$	3	8
q^7	1	0	0	$2\pi/M$	—	$\{14, 0, 0\}$	1	
	0	2	0	—	$2\pi/M$	$\{0, 2+12, 0\}$, $\{0, 4+10, 0\}$, $\{0, 6+8, 0\}$	3	
	0	0	2	—	$2\pi/M$	$\{0, 0, 2+12\}$, $\{0, 0, 4+10\}$, $\{0, 0, 6+8\}$	3	
	1	1	1	$4\pi/M$	$3\pi/M$	$\{4, 7, 3\}$, $\{4, 5, 5\}$, $\{4, 3, 7\}$ $\{6, 3, 5\}$, $\{6, 5, 3\}$, $\{8, 3, 3\}$	6	13
q^8	1	0	0	$2\pi/M$	—	$\{16, 0, 0\}$	1	
	0	2	0	—	$2\pi/M$	$\{0, 2+14, 0\}$, $\{0, 4+12, 0\}$, $\{0, 6+10, 0\}$	3	
	0	0	2	—	$2\pi/M$	$\{0, 0, 2+14\}$, $\{0, 0, 4+12\}$, $\{0, 0, 6+10\}$	3	
	1	1	1	$4\pi/M$	$3\pi/M$	$\{4, 9, 3\}$, $\{4, 7, 5\}$, $\{4, 5, 7\}$ $\{4, 3, 9\}$, $\{6, 7, 3\}$, $\{6, 5, 5\}$ $\{6, 3, 7\}$, $\{8, 5, 3\}$, $\{8, 3, 5\}$ $\{10, 3, 3\}$	10	17

[a] The minimum momenta $P^{ns}_{\min} = (\pi/M)(m_{ns} + m_{2s} + m_{-2s} + 1)$ and $P^{\pm}_{\min} = (\pi/M) \times (m_{ns} + (m_{2s} + m_{-2s})/2 + 1)$ are obtained from (3.15). The terms in $q^{1/24}b_4^0$ are obtained from (2.30c). Here $m_{2s} + m_{-2s}$ is even and $m_{ns} + m_{-2s} + (m_{2s} + m_{-2s})/2$ is odd. The macroscopic momentum shift is $\Delta P = 0$.

Further, the interchange $m^l_{2s} \leftrightarrow m^l_{-2s}$ leaves (3.14) invariant and from (2.20) gives $C \leftrightarrow -C$. Thus we need only consider $m^l_{2s} < m^l_{-2s}$ and find that this construction agrees with $q^{1/24} b^0_2$ of (2.30b).

In Table III we consider the case $m^r_{ns} = 1$, $m^r_{2s} = m^r_{-2s} = 0$ and compute to order q^8 the contribution made to Z in the channel $C/C_{\mathrm{GS}} = 1$ of the terms in (3.14) that involve only $P^{l,\alpha}_j$. From (2.7) and (2.20) we find that

$$m^l_{2s} + m^l_{-2s} \quad \text{even} \qquad \text{and} \qquad m^l_{ns} + m^l_{-2s} + (m^l_{2s} + m^l_{-2s})/2 \quad \text{odd} \tag{3.18}$$

We find that this agrees with $q^{1/24} b^0_4$ of (2.30c).

These above two equalities have been verified to order q^{200}.

From these constructions we can find expressions for b^0_2 and b^0_4 as we did above for b^0_0. We thus find that $q^{1/24} b^0_\alpha$ is given by (3.13) for $\alpha = 0, 2, 4$, where

$$\begin{array}{llll} \text{for } b^0_0 & m_{2s} + m_{-2s} \text{ is even} & \text{and} & m_{ns} + m_{-2s} + \dfrac{m_{2s} + m_{-2s}}{2} \text{ is even} \\ \text{for } b^0_2 & m_{2s} + m_{-2s} \text{ is odd} & \text{and} & m_{2s} < m_{-2s} \\ \text{for } b^0_4 & m_{2s} + m_{-2s} \text{ is even} & \text{and} & m_{ns} + m_{-2s} + \dfrac{m_{2s} + m_{-2s}}{2} \text{ is odd} \end{array} \tag{3.19}$$

This form for the branching functions agrees with the forms obtained in ref. 15.

We may now finally construct the complete $Q = 0$ contribution to Z by using (3.14)–(3.16) in (3.1) and summing over all m^r_α and m^l_α subject only to the restriction (2.17) written in the form

$$m^r_{2s} + m^r_{-2s} + m^l_{2s} + m^l_{-2s} \quad \text{even} \tag{3.20}$$

[We note that there is no restriction corresponding to (2.20), because both channels $C = \pm 1$ are considered in the sum.] It is easy then to see that the result consist of all the terms in (1.10) which involve b^0_0, b^0_2, and b^0_4, where we note (1) that the factor of 4 in front of $b^0_2(q)\, b^0_2(\bar{q})$ arises because of the symmetry under $m^r_{2s} \leftrightarrow m^r_{-2s}$ and $m^l_{2s} \leftrightarrow m^l_{-2s}$, and (2) terms like $b^0_0(q)\, b^0_2(\bar{q})$ and $b^0_4(q)\, b^0_2(\bar{q})$ are excluded by (3.20).

4. BRANCHING FUNCTIONS FOR $Q = 1$

The channel $Q = \pm 1$ is more complicated than the channel $Q = 0$ because, as seen in Section 2, the spectrum of excitations has four separate contributions. In ref. 17 these contributions are distinguished by the number m_{++} of $(++)$ pairs of roots and the number m_{-+} of $(-+)$ pairs of roots, where the sum rule is

$$m_{2s}+2m_{ns}+3m_{-2s}+m_{-+}+m_{++}=M-1 \tag{4.1}$$

We found that three cases occurred,

$$m_{-+}-m_{++}=1, 0, -1 \tag{4.2}$$

and when $m_{-+}=m_{++}$ the spectrum is twofold degenerate. We will thus extend the considerations of Section 3 by considering these three cases separately.

4.1. $m_{-+}-m_{++}=-1$

In this sector the total momentum is given by (2.15) with $P^0=P_{\mathrm{GS}}+\pi$ and there are $M-1$ single-particle states with $m_{ns}=1$.

We find for all three cases (4.2) from the previous study of completeness[17] that

$$P_j^{2s} \quad \text{takes} \quad M-1+m_{++}+m_{-2s} \quad \text{values} \tag{4.3a}$$

$$P_j^{-2s} \quad \text{takes} \quad m_{-2s}+m_{++}-1 \quad \text{values} \tag{4.3b}$$

and

$$P_j^{ns} \quad \text{takes} \quad m_{ns}+2m_{++}+2m_{-2s} \quad \text{values} \tag{4.3c}$$

In this present case we use $m_{-+}=m_{++}-1$ in (4.1) to write

$$2m_{++}=M-m_{2s}-2m_{ns}-3m_{-2s} \tag{4.4}$$

and thus (4.3) reduces to

$$\frac{3M}{2}-1-m_{ns}-\frac{m_{2s}+m_{-2s}}{2} \quad \text{values for } P_j^{2s} \tag{4.5a}$$

$$\frac{M}{2}-1-m_{ns}-\frac{m_{2s}+m_{-2s}}{2} \quad \text{values for} P_j^{-2s} \tag{4.5b}$$

and

$$M-m_{ns}-m_{2s}-m_{-2s} \quad \text{values for } P_j^{ns} \tag{4.5c}$$

where $m_{2s}+m_{-2s}$ is even. This will be the case if P_j^{α} satisfies

$$\frac{\pi}{M}\left(m_{ns}+\frac{m_{2s}+m_{-2s}}{2}+2\right) \leqslant P_j^{2s} \leqslant 3\pi-\frac{\pi}{M}\left(m_{ns}+\frac{m_{2s}+m_{-2s}}{2}+2\right) \tag{4.6a}$$

$$\frac{\pi}{M}\left(m_{ns}+\frac{m_{2s}+m_{-2s}}{2}+2\right) \leqslant P_j^{-2s} \leqslant \pi-\frac{\pi}{M}\left(m_{ns}+\frac{m_{2s}+m_{-2s}}{2}+2\right) \tag{4.6b}$$

and

$$\frac{\pi}{M}(m_{ns}+m_{2s}+m_{-2s}+1)\leqslant P_j^{ns}\leqslant 2\pi-\frac{\pi}{2}(m_{ns}+m_{2s}+m_{-2s}+1) \tag{4.6c}$$

where the spacing between allowed values of P_j^α is $2\pi/M$ and (1.4) holds. It may be verified that this choice of P_j^α exactly reproduces the momenta of Table 6 of ref. 17 for each allowed set of m_{2s}, m_{-2s}, and m_{ns}. Following the procedure of Section 3, we compute in Table IV the contribution these states will make to the partition function, where we use the linearized energies (3.6) and keep all P_j^α near zero.

4.2. $m_{-+}-m_{++}=1$

In this case $P^0=P_{\text{GS}}$ and there are $M-3$ single-particle states with $m_{ns}=1$. Furthermore, we find from (4.1) that

$$2m_{++}=M-2-m_{2s}-2m_{ns}-3m_{-2s} \tag{4.7}$$

and thus there are

$$\frac{3M}{2}-2-m_{ns}-\frac{m_{2s}+m_{-2s}}{2} \quad \text{values for } P_j^{2s} \tag{4.8a}$$

$$\frac{M}{2}-2-m_{ns}-\frac{m_{2s}+m_{-2s}}{2} \quad \text{values for } P_j^{-2s} \tag{4.8b}$$

and

$$M-2-m_{ns}-m_{2s}-m_{-2s} \quad \text{values for } P_j^{ns} \tag{4.8c}$$

where again $m_{2s}+m_{-2s}$ is even. This will be satisfied P_j^α satisfies

$$\frac{\pi}{M}\left(m_{ns}+\frac{m_{2s}+m_{-2s}}{2}+3\right)\leqslant P_j^{2s}\leqslant 3\pi-\frac{\pi}{M}\left(m_{ns}+\frac{m_{2s}+m_{-2s}}{2}+3\right) \tag{4.9a}$$

$$\frac{\pi}{M}\left(m_{ns}+\frac{m_{2s}+m_{-2s}}{2}+3\right)\leqslant P_j^{-2s}\leqslant \pi-\frac{\pi}{M}\left(m_{ns}+\frac{m_{2s}+m_{-2s}}{2}+3\right) \tag{4.9b}$$

$$\frac{\pi}{M}(m_{ns}+m_{2s}+m_{-2s}+3)\leqslant P_j^{ns}\leqslant 2\pi-\frac{\pi}{2}(m_{ns}+m_{2s}+m_{-2s}+3) \tag{4.9c}$$

where the spacing between allowed values of P_j^α is $2\pi/M$ and (1.4) holds. Again it may be verified that this choice of P_j^α exactly reproduces the momenta of Table 6 of ref. 17 for the allowed values of m_2, m_{-2s}, and m_{ns}. In Table V we compute the contribution these states make to the partition

Table IV. The Terms through Order q^8 in the Sector $m_{-+} - m_{++} = -1$ Constructed from the Rules of Section 4.1[a]

Order	m^l_{ns}	m^l_{2s}	m^l_{-2s}	$P^{ns}_{\min}$	$P^{\pm 2s}_{\min}$	$\{P^{ns}, P^{2s}, P^{-2s}\}$ (units of π/M)	States	Total
q^0	0	0	0	—	—	{0, 0, 0}	1	1
q^1	1	0	0	$2\pi/M$	—	{2, 0, 0}	1	1
q^2	1	0	0	$2\pi/M$	—	{4, 0, 0}	1	1
q^3	1	0	0	$2\pi/M$	—	{6, 0, 0}	1	
	0	1	1	—	$3\pi/M$	{0, 3, 3}	1	2
q^4	1	0	0	$2\pi/M$	—	{8, 0, 0}	1	
	0	1	1	—	$3\pi/M$	{0, 3, 5}, {0, 5, 3}	2	
	2	0	0	$3\pi/M$	—	{3 + 5, 0, 0}	1	
	0	2	0	—	$3\pi/M$	{0, 3 + 5, 0}	1	
	0	0	2	—	$3\pi/M$	{0, 0, 3 + 5}	1	6
q^5	1	0	0	$2\pi/M$	—	{10, 0, 0}	1	
	0	1	1	—	$3\pi/M$	{0, 3, 7}, {0, 5, 5}, {0, 7, 3}	3	
	2	0	0	$3\pi/M$	—	{3 + 7, 0, 0}	1	
	0	2	0	—	$3\pi/M$	{0, 3 + 7, 0}	1	
	0	0	2	—	$3\pi/M$	{0, 0, 3 + 7}	1	7
q^6	1	0	0	$2\pi/M$	—	{12, 0, 0}	1	
	0	1	1	—	$3\pi/M$	{0, 3, 9}, {0, 5. 7} {0, 7, 5}, {0, 9, 3}	4	
	2	0	0	$3\pi/M$	—	{3 + 9, 0, 0}, {5 + 7, 0, 0}	2	
	0	2	0	—	$3\pi/M$	{0, 3 + 9, 0}, {0, 5 + 7, 0}	2	
	0	0	2	—	$3\pi/M$	{0, 0, 3 + 7}, {0, 0, 5 + 7}	2	
	1	1	1	$4\pi\mu/M$	$4\pi/M$	{4, 4, 4}	1	12
q^7	1	0	0	$2\pi/M$	—	{14, 0, 0}	1	
	0	1	1	—	$3\pi/M$	{0, 3, 11}, {0, 5, 9}, {0, 7, 7}, {0, 9, 5}, {0, 11, 3}	5	
	2	0	0	$3\pi/M$	—	{3 + 11, 0, 0}, {5 + 9, 0, 0}	2	
	0	2	0	—	$3\pi/M$	{0, 3 + 11, 0}, {0, 5 + 9, 0}	2	
	0	0	2	—	$3\pi/M$	{0, 0, 3 + 11}, {0, 0, 5 + 9}	2	
	1	1	1	$4\pi/M$	$4\pi/M$	{4, 4, 6}, {4, 6, 4}, {6, 4, 4}	3	
	1	2	0	$4\pi/M$	$4\pi/M$	{4, 4 + 6, 0}	1	
	1	0	2	$4\pi/M$	$4\pi/M$	{4, 0, 4 + 6}	1	17
q^8	1	0	0	$2\pi/M$	—	{16, 0, 0}	1	
	0	1	1	—	$3\pi/M$	{0, 3, 13}, {0, 5, 11}, {0, 7, 9}, {0, 9, 7}, {0, 11, 5}, {0, 13, 3}	6	
	2	0	0	$3\pi/M$	—	{3 + 13, 0, 0}, {5 + 11, 0, 0}, {7 + 9, 0, 0}	3	
	0	2	0	—	$3\pi/M$	{0, 3 + 13, 0}, {0, 5 + 11, 0}, {0, 7 + 9, 0}	3	
	0	0	2	—	$3\pi/M$	{0, 0, 3 + 13}, {0, 0, 5 + 11}, {0, 0, 7 + 9}	3	
	1	1	1	$4\pi/M$	$4\pi/M$	{4, 4, 8}, {4, 8, 4}, {8, 4, 4}, {4, 6, 6}, {6, 4, 6}, {6, 6, 4}	6	
	1	2	0	$4\pi/M$	$4\pi/M$	{4, 4 + 8, 0}, {6, 4 + 6, 0}	2	
	1	0	2	$4\pi/M$	$4\pi/M$	{4, 0, 4 + 8}, {6, 0, 4 + 6}	2	26

[a] The minimum momenta $P^{ns}_{\min} = (\pi/M)(m_{ns} + m_{2s} + m_{-2s} + 1)$ and $P^{\pm 2s}_{\min} = (\pi/M) \times (m_{ns} + (m_{2s} + m_{-2s})/2 + 2)$ are obtained from (4.6). Here $m_{2s} + m_{-2s}$ is even and the macroscopic momentum shift is $\Delta P = \pi$.

function where the linearized energies (3.6) are used and all momenta are kept near zero.

4.3. $m-+=m++$

In this case $P^0 = P_{GS}$, there are $3M-4$ single-particles states with $m_{2s}=1$ and $M-4$ single-particle states with $m_{-2s}=1$. Furthermore,

$$2m_{++} = M - 1 - m_{2s} - 2m_{ns} - 3m_{-2s} \tag{4.10}$$

Table V. The Terms through Order q^8 in the Sector $m_{-+} - m_{++} = 1$ Constructed from the Rules of Section 4.2[a]

Order	m^l_{ns}	m^l_{2s}	m^l_{-2s}	$P^{ns}_{\min}$	$P^{\pm 2s}_{\min}$	$\{P^{ns}, P^{2s}, P^{-2s}\}$ (units of π/M)	States	Total
q^0	0	0	0	—	—	$\{0, 0, 0\}$	1	1
q^2	1	0	0	$4\pi/M$	—	$\{4, 0, 0\}$	1	1
q^3	1	0	0	$4\pi/M$	—	$\{6, 0, 0\}$	1	1
q^4	1	0	0	$4\pi/M$	—	$\{8, 0, 0\}$	1	
	0	1	1	—	$4\pi/M$	$\{0, 4, 4\}$	1	2
q^5	1	0	0	$4\pi/M$	—	$\{10, 0, 0\}$	1	
	0	1	1	—	$4\pi/M$	$\{0, 4, 6\}$, $\{0, 6, 4\}$	2	
	0	2	0	—	$4\pi/M$	$\{0, 4+6, 0\}$	1	
	0	0	2	—	$4\pi/M$	$\{0, 0, 4+6\}$	1	5
q^6	1	0	0	$4\pi/M$	—	$\{12, 0, 0\}$	1	
	0	1	1	—	$4\pi/M$	$\{0, 4, 8\}$, $\{0, 6, 6\}$, $\{0, 8, 4\}$	3	
	0	2	0	—	$4\pi/M$	$\{0, 4+8, 0\}$	1	
	0	0	2	—	$4\pi/M$	$\{0, 0, 4+8\}$	1	
	2	0	0	$5\pi/M$	—	$\{5+7, 0, 0\}$	1	7
q^7	1	0	0	$4\pi/M$	—	$\{14, 0, 0\}$	1	
	0	1	1	—	$4\pi/M$	$\{0, 4, 10\}$, $\{0, 6, 8\}$, $\{0, 8, 6\}$, $\{0, 10, 4\}$	4	
	0	2	0	—	$4\pi/M$	$\{0, 4+10, 0\}$, $\{0, 6+8, 0\}$,	2	
	0	0	2	—	$4\pi/M$	$\{0, 0, 4+10\}$, $\{0, 0, 6+8\}$	2	
	2	0	0	$5\pi/M$	—	$\{5+9, 0, 0\}$	1	10
q^8	1	0	0	$4\pi/M$	—	$\{16, 0, 0\}$	1	
	0	1	1	—	$4\pi/M$	$\{0, 4, 12\}$, $\{0, 6, 10\}$, $\{0, 8, 8\}$, $\{0, 10, 6\}$, $\{0, 12, 4\}$	5	
	0	2	0	—	$4\pi/M$	$\{0, 4+12, 0\}$, $\{0, 6+10, 0\}$,	2	
	0	0	2	—	$4\pi/M$	$\{0, 0, 4+12\}$, $\{0, 0, 6+10\}$	2	
	2	0	0	$5\pi/M$	—	$\{5+11, 0, 0\}$, $\{7+9, 0, 0\}$	2	
	1	1	1	$6\pi/M$	$5\pi/M$	$\{6, 5, 5\}$	1	13

[a] The minimum momenta $P^{ns}_{\min} = (\pi/M)(m_{ns} + m_{2s} + m_{-2s} + 3)$ and $P^{\pm 2s}_{\min} = (\pi/M) \times (m_{ns} + (m_{2s} + m_{-2s})/2 + 3)$ are obtained from (4.9). Here $m_{2s} + m_{-2s}$ is even and the macroscopic momentum shift is $\Delta P = 0$.

where now $m_{2s}+m_{-2s}$ is odd. In this case there is a double degeneracy and we find from ref. 17 that there are

$$2\times\left(\frac{3M}{2}-1-m_{ns}-\frac{m_{2s}+m_{-2s}+1}{2}\right) \qquad \text{values for } P_j^{2s} \tag{4.11a}$$

$$2\times\left(\frac{M}{2}-1-m_{ns}-\frac{m_{2s}+m_{-2s}+1}{2}\right) \qquad \text{values for } P_j^{-2s} \tag{4.11b}$$

and

$$2\times(M-1-m_{ns}-m_{2s}-m_{-2s}) \qquad \text{values for } P_j^{ns} \tag{4.11c}$$

Now in order to get a formula for the momentum which respects (4.11) we must consider two subcases. Either P_j^{α} satisfy

$$\frac{\pi}{M}\left(m_{ns}+\frac{m_{2s}+m_{-2s}+1}{2}+1\right) \leqslant P_j^{2s} \leqslant 3\pi-\frac{\pi}{M}\left(m_{ns}+\frac{m_{2s}+m_{-2s}+1}{2}+3\right) \tag{4.12a}$$

$$\frac{\pi}{M}\left(m_{ns}+\frac{m_{2s}+m_{-2s}+1}{2}+1\right) \leqslant P_j^{-2s} \leqslant \pi-\frac{\pi}{M}\left(m_{ns}+\frac{m_{2s}+m_{-2s}+1}{2}+3\right) \tag{4.12b}$$

$$\frac{\pi}{M}(m_{ns}+m_{2s}+m_{-2s}+1) \leqslant P_j^{ns} \leqslant 2\pi-\frac{\pi}{M}(m_{ns}+m_{2s}+m_{-2s}+3) \tag{4.12c}$$

or P_j^{α} satisfy

$$-3\pi+\frac{\pi}{M}\left(m_{ns}+\frac{m_{2s}+m_{-2s}+1}{2}+3\right) \leqslant P_j^{2s} \leqslant -\frac{\pi}{M}\left(m_{ns}+\frac{m_{2s}+m_{-2s}+1}{2}+1\right) \tag{4.13a}$$

$$-\pi+\frac{\pi}{M}\left(m_{ns}+\frac{m_{2s}+m_{-2s}+1}{2}+3\right) \leqslant P_j^{-2s} \leqslant -\frac{\pi}{M}\left(m_{ns}+\frac{m_{2s}+m_{-2s}+1}{2}+1\right) \tag{4.13b}$$

$$-2\pi + \frac{\pi}{M}(m_{ns} + m_{2s} + m_{-2s} + 3)$$

$$\leqslant P_j^{ns} \leqslant -\frac{\pi}{M}(m_{ns} + m_{2s} + m_{-2} + 1) \qquad (4.13c)$$

However, now instead of the total momentum P being given in terms of P_j^α by (2.15), we must introduce a shift of order $1/M$ [which is

Table VI. The Terms through Order q^8 in the Sector $m_{-+} = m_{++}$ with the Macroscopic Momentum $\Delta P = 0$ Constructed from theRules of Section 4.3[a]

Order	m_{ns}^l	m_{2s}^l	m_{-2s}^l	$P_{\min}^{ns}$	$P_{\min}^{\pm 2s}$	$\{P^{ns}, P^{2s}, P^{-2s}\}$ (units of π/M)	Shift	States	Total
q^1	0	1	0	—	$2\pi/M$	$\{0, 2, 0\}$	0	1	2
q^2	0	1	0	—	$2\pi/M$	$\{0, 4, 0\}$	0	1	2
q^3	0	1	0	—	$2\pi/M$	$\{0, 6, 0\}$	0	1	
	1	1	0	$3\pi/M$	$3\pi/M$	$\{3, 3, 0\}$	0	1	4
q^4	0	1	0	—	$2\pi/M$	$\{0, 8, 0\}$	0	1	
	1	1	0	$3\pi/M$	$3\pi/M$	$\{3, 5, 0\}$, $\{5, 3, 0\}$	0	2	6
q^5	0	1	0	—	$2\pi/M$	$\{0, 10, 0\}$	0	1	
	1	1	0	$3\pi/M$	$3\pi/M$	$\{3, 7, 0\}$, $\{5, 5, 0\}$, $\{7, 3, 0\}$	0	3	8
q^6	0	1	0	—	$2\pi/M$	$\{0, 12, 0\}$	0	1	
	1	1	0	$3\pi/M$	$3\pi/M$	$\{3, 9, 0\}$, $\{5, 7, 0\}$			
						$\{7, 5, 0\}$, $\{9, 3, 0\}$	0	4	
	0	2	1	—	$3\pi/M$	$\{0, 3+5, 3\}$	π/M	1	12
q^7	0	1	0	—	$2\pi/M$	$\{0, 14, 0\}$	0	1	
	1	1	0	$3\pi/M$	$3\pi/M$	$\{3, 11, 0\}$, $\{5, 9, 0\}$			
						$\{7, 7, 0\}$, $\{9, 5, 0\}$, $\{11, 3, 0\}$	0	5	
	0	2	1	—	$3\pi/M$	$\{0, 3+7, 3\}$, $\{0, 3+5, 5\}$	π/M	2	
	2	1	0	$4\pi/M$	$4\pi/M$	$\{4+6, 4, 0\}$	0	1	18
q^8	0	1	0	—	$2\pi/M$	$\{0, 16, 0\}$	0	1	
	1	1	0	$3\pi/M$	$3\pi/M$	$\{3, 13, 0\}$, $\{5, 11, 0\}$, $\{7, 9, 0\}$			
						$\{9, 7, 0\}$, $\{11, 5, 0\}$, $\{13, 3, 0\}$	0	6	
	0	2	1	—	$3\pi/M$	$\{0, 3+9, 3\}$, $\{0, 5+7, 3\}$			
						$\{0, 3+7, 5\}$, $\{0, 3+5, 7\}$	π/M	4	
	2	1	0	$4\pi/M$	$4\pi/M$	$\{4+8, 4, 0\}$, $\{4+6, 6, 0\}$	0	2	
	0	3	0	—	$3\pi/M$	$\{0, 3+5+7, 0\}$	π/M	1	28

[a] The minimum momenta $P_{\min}^{ns} = (\pi/M)(m_{ns} + m_{2s} + m_{-2s} + 1)$ and $P_{\min}^{\pm 2s} = (\pi/M) \times (m_{ns} + (m_{2s} + m_{-2s} + 1)/2 + 1)$ are obtained from (4.12). Here $m_{2s} + m_{-2s}$ is odd and only $m_{2s} > m_{-2s}$ are explicitly shown.

permissible because (2.15) is only derived to order one] and write that when (4.12) holds

$$P=P^0+\sum_{\alpha=2s,-2s,ns}\sum_{j=1}^{m_\alpha} P_j^\alpha+\frac{\pi}{M}\left(\frac{m_{2s}+m_{-2s}-1}{2}\right) \tag{4.14a}$$

and when (4.13) holds

$$P=P^0+\sum_{\alpha=2s,-2s,ns}\sum_{j=1}^{m_\alpha} P_j^\alpha-\frac{\pi}{M}\left(\frac{m_{2s}+m_{-2s}-1}{2}\right) \tag{4.14b}$$

It can be verified that the momenta computed from these rules agree with the momenta of Table 6 of ref. 17 for the allowed values of m_{2s}, m_{-2s}, and m_{ns}.

Corresponding to this momentum shift there is an energy shift as well. Thus in Table VI we compute the contribution to the partition function of those states obtained from (4.12) with P_j^α near zero using the linearized energies (3.6) and subtracting the shift $(\pi/M)((m_{2s}+m_{-2s}-1)/2)$. We note that the macroscopic momenta of these states are near P_{GS}. In Table VII we compute the contribution to the partition function from those states obtained from (4.13) with P_j^{2s} near -3π, P_j^{-2s} near $-\pi$, and P_j^{ns} near -2π, where we add a shift $(\pi/M)((m_{2s}+m_{-2s}-1)/2)$ to the energy. The macroscopic momentum of these states in $P_{\text{GS}}+\pi$. In both cases, due to the symmetry under $2s \leftrightarrow -2s$, only the states with $m_{2s}<m_{-2s}$ are shown.

4.4. Branching Functions

We may now obtain the formulas for the two branching functions of $Q=1$, namely b_2^0 and b_2^2, by combining the results of the three preceding subsections with the same macroscopic momentum.

Consider first the case where the macroscopic momentum is P_{GS}. This is obtained from the $m_{-+}-m_{++}=1$ states of Table V and the $m_{-+}=m_{++}$ states of Table VI. In Table VIII compute the sum of these two contributions and see that it is identical with the corresponding terms in $q^{-1/3}q^{1/24}b_0^2$ of (2.30d).

The other case has the macroscopic momentum of $P_{\text{GS}}+\pi$. This is obtained from the $m_{-+}-m_{++}=-1$ states of Table IV and the $m_{-+}=m_{++}$ states of Table VII. In Table IX we compute the sum of these two contributions and see that it is identical with the corresponding terms in $q^{-1/12}q^{1/24}b_2^2$ of (2.30e).

As in $Q=0$, these identities have been verified to order q^{200}.

We may now use the above construction to obtain formulas for the branching functions b_0^2 and b_2^2 by following exactly the same procedure used in Section 3. This we find

$$q^{-1/3}q^{1/24}b_0^2$$

$$= \sum_{m_{ns}=0}^{\infty} \sum_{m_{2s}=0}^{\infty} \sum_{\substack{m_{-2s}=0 \\ m_{2s}+m_{-2s}\ \text{even}}}^{\infty} \frac{q^{m_{ns}(m_{ns}+1)/2}}{(q)_{m_{ns}}} \frac{q^{m_{2s}(m_{2s}+1)/2}}{(q)_{m_{2s}}} \frac{q^{m_{-2s}(m_{-2s}+1)/2}}{(q)_{m_{-2s}}}$$

$$\times q^{(m_{ns}/2)(m_{ns}+m_{2s}+m_{-2s}+1)} q^{((m_{2s}+m_{-2s})/2)(m_{ns}+(m_{2s}+m_{-2s})/2)+1)}$$

$$+ \sum_{m_{ns}=0}^{\infty} \sum_{m_{2s}=0}^{\infty} \sum_{\substack{m_{-2s}=0 \\ m_{2s}+m_{-2s}\ \text{odd}}}^{\infty} \frac{q^{m_{ns}(m_{ns}+1)/2}}{(q)_{m_{ns}}} \frac{q^{m_{2s}(m_{2s}+1)/2}}{(q)_{m_{2s}}} \frac{q^{m_{-2s}(m_{-2s}+1)/2}}{(q)_{m_{-2s}}}$$

$$\times q^{(m_{ns}/2)(m_{ns}+m_{2s}+m_{-2s}-1)} q^{((m_{2s}+m_{-2s})/2)(m_{ns}+(m_{2s}+m_{-2s})/2)} q^{-1/4}$$

$$= \sum_{m_{ns}=0}^{\infty} \sum_{m_{2s}=0}^{\infty} \sum_{\substack{m_{-2s}=0 \\ m_{2s}+m_{-2s}\ \text{even}}}^{\infty}$$

$$\times \frac{q^{(4m_{ns}^2+3m_{2s}^2+3m_{-2s}^2+4m_{ns}m_{2s}+4m_{ns}m_{-2s}+2m_{2s}m_{-2s}+4m_{ns}+4m_{2s}+4m_{-2s}-1)/4}}{(q)_{m_{ns}}(q)_{m_{2s}}(q)_{m_{-2s}}}$$

$$+ \sum_{m_{ns}=0}^{\infty} \sum_{m_{2s}=0}^{\infty} \sum_{\substack{m_{-2s}=0 \\ m_{2s}+m_{-2s}\ \text{odd}}}^{\infty}$$

$$\times \frac{q^{(4m_{ns}^2+3m_{2s}^2+3m_{-2s}^2+4m_{ns}m_{2s}+4m_{ns}m_{-2s}+2m_{2s}m_{-2s}+2m_{2s}+2m_2+m_{-2s})/4}}{(q)_{m_{ns}}(q)_{m_{2s}}(q)_{m_{-2s}}} \tag{4.15a}$$

and

$$q^{-1/12}q^{1/24}b_2^2$$

$$= \sum_{m_{ns}=0}^{\infty} \sum_{m_{2s}=0}^{\infty} \sum_{\substack{m_{-2s}=0 \\ m_{2s}+m_{-2s}\ \text{even}}}^{\infty} \frac{q^{m_{ns}(m_{ns}+1)/2}}{(q)_{m_{ns}}} \frac{q^{m_{2s}(m_{2s}+1)/2}}{(q)_{m_{2s}}} \frac{q^{m_{-2s}(m_{-2s}+1)/2}}{(q)_{m_{-2s}}}$$

$$\times q^{(m_{ns}/2)(m_{ns}+m_{2s}+m_{-2s}+1)} q^{((m_{2s}+m_{-2s})/2)(m_{ns}+(m_{2s}+m_{-2s})/2)}$$

$$+ \sum_{m_{ns}=0}^{\infty} \sum_{m_{2s}=0}^{\infty} \sum_{\substack{m_{-2s}=0 \\ m_{2s}+m_{-2s}\ \text{odd}}}^{\infty} \frac{q^{m_{ns}(m_{ns}+1)/2}}{(q)_{m_{ns}}} \frac{q^{m_{2s}(m_{2s}+1)/2}}{(q)_{m_{2s}}} \frac{q^{m_{-2s}(m_{-2s}+1)/2}}{(q)_{m_{-2s}}}$$

$$\times q^{(m_{ns}/2)(m_{ns}+m_{2s}+m_{-2s}+1)} q^{((m_{2s}+m_{-2s})/2)(m_{ns}+(m_{2s}+m_{-2s})/2+1)} q^{-1/4}$$

$$= \sum_{m_{ns}=0}^{\infty} \sum_{m_{2s}=0}^{\infty} \sum_{\substack{m_{-2s}=0 \\ m_{2s}+m_{-2s}\ \text{even}}}^{\infty}$$

$$\times \frac{q^{(4m_{ns}^2+3m_{2s}^2+3m_{-2s}^2+4m_{ns}m_{2s}+4m_{ns}m_{-2s}+2m_{2s}m_{-2s}+2m_{2s}+2m_{-2s})/4}}{(q)_{m_{ns}}(q)_{m_{2s}}(q)_{m_{-2s}}}$$

$$+ \sum_{m_{ns}=0}^{\infty} \sum_{m_{2s}=0}^{\infty} \sum_{\substack{m_{-2s}=0 \\ m_{2s}+m_{-2s}\ \text{odd}}}^{\infty}$$

$$\times \frac{q^{(4m_{ns}^2+3m_{2s}^2+3m_{-2s}^2+4m_{ns}m_{2s}+4m_{ns}m_{-2s}+2m_{2s}m_{-2s}+4m_{ns}+4m_{2s}+4m_{-2s}+1)/4}}{(q)_{m_{ns}}(q)_{m_{2s}}(q)_{m_{-2s}}} \tag{4.15b}$$

Table VII. The Terms through Order q^8 in the Sector $m_{-+}=m_{++}$ with the Macroscopic Momentum $\Delta P=\pi$ Constructed from the Rules of Section 4.3[a]

Order	m^l_{ns}	m^l_{2s}	m^l_{-2s}	$P^{ns}_{\min}$	$P^{\pm 2s}_{\min}$	$\{P^{ns}, P^{2s}, P^{-2s}\}$ (units of π/M)	Shift	States	Total
q^2	0	1	0	—	$4\pi/M$	$\{0, 4, 0\}$	0	1	2
q^3	0	1	0	—	$4\pi/M$	$\{0, 6, 0\}$	0	1	2
q^4	0	1	0	—	$4\pi/M$	$\{0, 8, 0\}$	0	1	2
q^5	0	1	0	—	$4\pi/M$	$\{0, 10, 0\}$	0	1	
	1	1	0	$5\pi/M$	$5\pi/M$	$\{5, 5, 0\}$	0	1	4
q^6	0	1	0	—	$4\pi/M$	$\{0, 12, 0\}$	0	1	
	1	1	0	$5\pi/M$	$5\pi/M$	$\{5, 7, 0\}$, $\{7, 5, 0\}$	0	2	6
q^7	0	1	0	—	$4\pi/M$	$\{0, 14, 0\}$	0	1	
	1	1	0	$5\pi/M$	$5\pi/M$	$\{5, 9, 0\}$, $\{7, 7, 0\}$, $\{9, 5, 0\}$	0	3	8
q^8	0	1	0	—	$4\pi/M$	$\{0, 16, 0\}$	0	1	
	1	1	0	$5\pi/M$	$5\pi/M$	$\{5, 11, 0\}$, $\{7, 9, 0\}$ $\{9, 7, 0\}$, $\{11, 5, 0\}$	0	4	
	0	2	1	—	$5\pi/M$	$\{0, 5+7, 5\}$	$-\pi/M$	1	12

[a] The minimum momenta $P^{ns}_{\min} = (\pi/M)(m_{ns} + m_{2s} + m_{-2s} + 3)$ and $P^{\pm 2s}_{\min} = (\pi/M) \times (m_{ns} + (m_{2s} + m_{-2s} + 1)/2 + 3)$ are obtained from (4.13). Here $m_{2s} + m_{-2s}$ is odd and only $m_{2s} > m_{-2s}$ are explicitly shown.

Table VIII. The Terms through Order q^8 in the Sum of $m_{-+} - m_{++} = 1$ and the $\Delta P = 0$ Term of $m_{-+} = m_{++}$[a].

Order	$m_{-+} - m_{++} = 1$	$m_{-+} - m_{++} = 0$	Total	$q^{-1/3} q^{1/24} b_0^2$
q^0	1	0	1	1
q^1	0	2	2	2
q^2	1	2	3	3
q^3	1	4	5	5
q^4	2	6	8	8
q^5	5	8	13	13
q^6	7	12	19	19
q^7	10	18	28	28
q^8	13	28	41	41

[a] These are compared with the terms in $q^{-1/3} q^{1/24} b_0^2$ of (2.30d).

Table IX. The Terms through Order q^8 in the Sum of $m_{-+} - m_{++} = -1$ and the $\Delta P = \pi$ Term of $m_{-+} = m_{++}$[a].

Order	$m_{-+} - m_{++} = 1$	$m_{-+} - m_{++} = 0$	Total	$q^{-1/12} q^{1/24} b_0^2$
q^0	1	0	1	1
q^1	1	0	1	1
q^2	1	2	3	3
q^3	2	2	4	4
q^4	6	2	8	8
q^5	7	4	11	11
q^6	12	6	18	18
q^7	17	8	25	25
q^8	26	12	38	38

[a] These are compared with the terms in $q^{-1/12} q^{1/24} b_2^2$ of (2.30e).

Unlike the case of $Q = 0$, these forms for the branching functions are not of the form of ref. 15.

4.5. Partition Function

Finally we need to compute the complete partition in the $Q = 1$ channel. Now in each of the four cases considered above there are both left and right excitations m_α^r and m_α^l. Denote the four sums obtained above for all $m_\alpha^r = 0$ by S^- for $m_{-+} - m_{++} = -1$, by S^+ for $m_{-+} - m_{++} = 1$, by S^0 for $m_{-+} = m_{++}$ with $\Delta P = 0$, and by S^π for $m_{-+} = m_{++}$ with $\Delta P = \pi$, To consider the general case with both m_α^r and m_α^l both nonzero, we follow the procedure of Section 3 and consider momentum restrictions for the tight and left movers separately.

Consider first $m_{-+} - m_{++} = -1$. Then, because of the restriction that $m_{2s} + m_{-2s} = m_{2s}^r + m_{+2s}^r + m_{-2s}^l + m_{-2s}^l$ must be even, we see that $m_{2s}^r + m_{-2s}^r$ and $m_{2s}^l + m_{-2s}^l$ must be even or odd together. If both terms are even, the contribution made to the partition function is $S^-(q)\, S^-(\bar{q})$ and if both terms are odd and the momenta are shifted properly, the contribution is $S^\pi(q)\, S^\pi(\bar{q})$.

The contribution from $m_{-+} - m_{++} = 1$ is similar. Again $m_{2s}^r + m_{-2s}^r$ and $m_{2s}^l + m_{-2s}^l$ must be even or odd together. The even terms contribute $S^+(q)\, S^+(\bar{q})$ and the odd terms contribute $S^0(q)\, S^0(\bar{q})$.

Finally there are the terms from $m_{++} = m_{-+}$. Here $m_{2s}^r + m_{-2s}^r + m_{2s}^l + m_{-2s}^l$ is odd, thus $m_{2s}^r + m_{-2s}^r$ is even (odd) and $m_{2s}^l + m_{-2s}^l$ is odd (even). The ends where $m_{2s}^{r,l} + m_{-2s}^{r,l}$ are odd produce the terms S^0 and S^π and if the momentum shift is properly accounted for, the ends with

$m_{2s}^{r,l}+m_{-2s}^{r,l}$ even produce S^+ and S^-. Thus, the sector $m_{++}=m_{-+}$ produces cross terms like $S^0(q)\,S^+(\bar{q})$ and $S^\pi(q)\,S^-(\bar{q})$, and hence the desired form $b_2^2(q)\,b_2^2(\bar{q})+b_0^2(q)\,b_0^2(\bar{q})$ in (1.10) is obtained.

5. DISCUSSION

Many mathematical points of the foregoing computations remain to be clarified, such as: (1) a direct proof of the equivalence of the forms (3.13), (3.19), and (4.16) with (2.25), (2.27), and the forms of Appendix B; (2) the obtaining of our results directly from the functional equations[16,21–23] without recourse to the study of the completeness rules.[17] However, the major feature of the results of Sections 3 and 4 is that they directly relate the concept of branching function with that of quasiparticle. This provides insight into the physics of the model, which we will discuss here in detail.

5.1. Infrared Momentum Restrictions

All the eigenvalue spectra computed here have, up to a possible constant, the additive quasiparticle form (1.2) of

$$E_{\text{ex}}-E_{\text{GS}}=\sum_{\alpha,\,\text{rules}} e_\alpha(P_j) \tag{5.1}$$

where two important features of the rules govern the combination of energy levels. The first is the Fermi exclusion property:

$$P_i^\alpha \neq P_j^\alpha \qquad \text{for} \quad i\neq j \tag{5.2}$$

This rule in conjunction with the quasiparticle form (5.1) is often used to say that the quasiparticles are fermions.

If the momenta P_j^α were such that

$$P_j^\alpha=\frac{2\pi j}{M} \tag{5.3}$$

with j an integer (or possibly half-integer), these quasiparticle energies would indeed be identical with those of a genuine free Fermi gas. However, the momentum set out of which the P_j are chosen is not (5.3). Instead, the momenta are subject to the restrictions (3.16), (4.6), (4.13), and (4.14). These all share the feature that there is a depletion of the number of allowed momenta near the values of P where $e_\alpha(P)=0$, which depends on the number of quasiparticles in the state. This is an intrinsic many-body effect which cannot be modeled by an effective change in some one- or two-body property.

More generally we may consider n species of fermionic quasiparticles which have a spacing of allowed values of $2\pi/M$ and a set of infrared momentum restrictions

$$\frac{\pi}{M}\left(\sum_{\beta} N_{\alpha,\beta} m_\beta + 1\right) \leqslant P_j^\alpha \tag{5.4}$$

This set of infrared quasiparticle momentum restrictions gives results which agree with the group-theoretic construction of ref. 15 for the branching functions of the Z_{n+1} parafermionic conformal field theories,[26,27] where $(N_{\alpha,\beta} + \delta_{\alpha,\beta})/2$ is the inverse Cartan matrix of A_n. Moreover, it can be shown[34] that a similar construction with $N_{\alpha,\beta}$ given by the inverse Cartan matrix of the D_n and E_n simply laced Lie algebras gives branching functions of the corresponding conformal field theories. Thus the infrared momentum restrictions (5.4) are a general mechanism that can lead to central charges being different from integer or half-integer.

The phenomena characterized by the momentum restrictions (5.4) have, at least on the face of it, nothing to do with integrability, Virasoro algebra, modular invariance, or any other symmetry algebra. In this respect the restrictions (5.4) share the feature of generality with Haldane's[35] definition of fractional statistics. However, the definition of ref. 35 differs from the infrared momentum restrictions (5.4) by relying on the finiteness of the Hilbert space. This can only be achieved by imposing an ultraviolet as well as an infrared cutoff on the problem, whereas, by the very name, infrared momentum restrictions exist without an ultraviolet cutoff. The essential feature of ref. 35, however, is the abandoning of a second-quantized description of the excitations in the system and this is certainly a key property of the effects described above.

5.2. Specific Heat

One of the powerful results of conformal field theory is the prediction[36,37] that the specific heat C for a system with periodic boundary conditions is given in terms of the central charge and the velocity of sound as $T \sim 0$ as

$$C \sim \frac{\pi k_B^2 c}{3v} T \tag{5.5}$$

The original derivations are based on conformal invariance. We discuss here the relation of (5.5) with the momentum restrictions (5.4).

By definition the bulk free energy per site is obtained from the partition function

$$Z = \mathrm{Tr}\, e^{-H/k_B T} \tag{5.6}$$

as

$$f = -k_B T \lim_{M \to \infty} \frac{1}{M} \ln Z \tag{5.7}$$

where

$$T > 0 \quad \text{is fixed} \qquad \text{and} \qquad M \to \infty \tag{5.8}$$

Then the specific heat is

$$C = -T \frac{\partial^2 f}{\partial T^2} \tag{5.9}$$

To obtain the leading behavior as $T \to 0$ of the specific heat it is sufficient to restrict attention to the low-lying (order-one or smaller) excitations of H over the ground state. These obey the quasiparticle form (5.1). Hence specific heats are commonly evaluated with formulas involving single-particle levels $e(P)$.

This argument, however, is not complete, as is apparent in the observation that any energy level with $e(P) > 0$ as $M \to \infty$ will contribute only a term exponentially small in T to the specific heat. Thus the order-one excitations do not contribute to the linear term (5.5). Instead it is the levels that have the property that $\lim_{M \to \infty} e(P) = 0$ which contribute to the leading behavior.

The partition function in the conformal field theory of one-dimensional quantum spin chains is computed in the limit

$$M \to \infty, \qquad T \to 0 \qquad \text{with } MT \text{ fixed} \tag{5.10}$$

This is not the same as the limit (5.8) which defines the specific heat. However, if no additional length scale appears in the system, it is expected that the behavior of the specific heat computed using the prescription (5.10) where $q = e^{-2\pi v/M k_B T}$ is fixed will agree when $q \to 1$ with the $T \to 0$ behavior computed using the prescription (5.8) for systems obeying the infrared momentum restrictions (5.4). The $q \to 1$ behavior of Z can be computed directly from the expression of the branching functions as infinite series in q.[38] Since these series are a direct consequence of the infrared momentum restrictions, the influence of the many-body effects is apparent.

These momentum restrictions can only be seen by imposing an explicit cutoff on the problem.

The above argument has relied only on a one-length-scale scaling argument and is valid for any set of momentum restrictions $N_{\alpha\beta}$ in (5.4). Moreover, the argument has not identified the coefficient c of (5.5) with the central charge which is obtained from the $q \to 0$ behavior of the branching functions. However, for the cases where $N_{\alpha\beta}$ is given by an inverse Cartan matrix, the $q \to 0$ and the $q \to 1$ behaviors are related by the property of modular covariance, which states that if we define τ as $q = e^{2\pi i\tau}$, then for a set of branching functions $b_k(\tau)$

$$b_k(-1/\tau) = \sum_l M_{k,l} b_l(\tau) \tag{5.11}$$

where the functions $M_{k,l}$ are independent of τ. Thus the $q \to 1$ behavior of the branching functions is given in terms of the $q \to 0$ behavior. This behavior is always $q^{-c/24 + h_k}$, where h_k is the conformal dimension and c is determined from the finite-size correction of the ground-state energy of (3.2). The right-hand side of (5.11) will thus be dominated by the term with the smallest h_k. In particular, if the smallest h_k is zero, the formula (5.5) results.

For systems described by conformal field theory, the partition function is modular invariant, and the branching functions modular covariant. This property allows us to compute the specific heat from the $q \to 0$ behavior of the branching functions. However, from the point of view of the momentum restrictions (5.4), this property is not necessary, and the specific heat can be computed directly from the $q \to 1$ behavior. It is an unsolved problem to determine directly from (5.4) which momentum restriction rules $N_{\alpha\beta}$ will lead to the modular covariance property (5.11).

5.3. Oscillations

One of the striking features of Sections 3 and 4 is the fact that the branching functions b_2^0 (with conformal dimension 3/4) and b_2^2 (with conformal dimension 1/12) are obtained with a macroscopic (order-one) momentum shift from the ground state of $\Delta P = \pi$. This results from the feature found in ref. 13 that the energies $e_{2s}(P)$ $[e_{-2s}(P)]$ vanish at 3π and π. These macroscopic momentum shifts are expected to give rise to oscillatory contributions to the correlation functions of the primary operators of b_2^0 and b_2^2. On the lattice the oscillatory term should be $(-1)^N$, where N is the separation of the operators. These microscopic oscillations are perhaps unusual for conformal field theories, but are in fact expected if we make the observation that the antiferromagnetic 3-state Potts model lies deep inside the incommensurate phase of the chiral Potts model,[16,39] which is characterized by oscillatory correlations.[40,41]

5.4. Lee–Yang Edge

An interesting property of the branching functions (3.13) and (4.16) obtains if we consider only the terms where $m_{2s} = m_{-2s} = 0$, where the sums reduce to the two sums

$$S_0 = \sum_{m_{ns}=0}^{\infty} \frac{q^{m_{ns}^2}}{(q)_{m_{ns}}} \tag{5.12a}$$

and

$$S_1 = \sum_{m_{ns}=0}^{\infty} \frac{q^{m_{ns}(m_{ns}+1)}}{(q)_{m_{ns}}} \tag{5.12b}$$

These are the famous sums of Rogers–Ramanujan.[42,44] They become modular functions if we multiply by powers of q and consider

$$c_{1,3}(\tau) = q^{-1/60} S_0 \qquad \text{and} \qquad c_{1,1}(\tau) = q^{11/60} S_1 \tag{5.13}$$

which, in fact, are shown in refs. 42–44 to be the two characters of the nonunitary minimal model[2] with $p = 5$ and $p' = 2$

$$c_{r,s}(q) = \frac{q^{-1/24}}{Q(q)} \sum_{n=-\infty}^{\infty} \{q^{(2npp' + rp - sp')^2/4pp'} - q^{(2npp' + rp + sp')^2/4pp'}\} \tag{5.14}$$

This model has been identified[18,19] with the field theory that describes the behavior at the Lee–Yang edge of the Ising model.[20] Thus there is a sense in which we may say that the field theory for the Lee–Yang edge is obtained by adding a perturbation to the 3-state antiferromagnetic Potts chain which makes the $\pm 2s$ excitations massive without affecting the ns excitations.

It is also interesting to note that the $q \to 1$ behavior of the sums (5.12) can be calculated directly without recourse to the modular transformation (5.11) by a method that makes contact with dilogarithms. As an example, we consider explicitly S_0, which we write in the form obtained directly from (3.10),

$$S_0 = \sum_{m=0} \sum_{n=0} P_d(m, n) q^b q^{(m/2)(m-1)} \tag{5.15}$$

(where the subscript *ns* has been dropped for simplicity). To study the limit $q \to 1$, we first use the integral representation

$$\sum_{n=0}^{\infty} P_d(m, n) q^n = \frac{1}{2\pi i} \oint \frac{dz}{z^{m+1}} \prod_{l=1}^{\infty} (1 + zq^l) \tag{5.16}$$

in the sum (5.15). We note that (5.16) vanishes if $m < 0$. Thus we extend the lower limit of the sum over m from zero to $-\infty$ and interchange the summation and integration to obtain

$$S_0 = \frac{1}{2\pi i} \oint \frac{dz}{z} \prod_{l=1}^{\infty} (1 + zq^l) \sum_{m=-\infty}^{\infty} q^{(m/2)(m-1)} z^{-m} \tag{5.17}$$

The sum over m is expressed in terms of the Jacobi theta function[45]

$$\theta_2(v, q) = \sum_{m=-\infty}^{\infty} q^{(m-1/2)^2} e^{i\pi(2m-1)v} \tag{5.18}$$

as

$$\sum_{m=-\infty}^{\infty} q^{(m/2)(m-1)} z^{-m} = q^{-1/8} z^{1/2} \theta_2(v, q^{1/2}) \tag{5.19}$$

with

$$e^{i\pi v} = z^{-1/2} \tag{5.20}$$

and hence

$$S_0 = \frac{1}{2\pi i} \oint \frac{dz}{z} \prod_{l=1}^{\infty} (1 + zq^l) q^{-1/8} z^{1/2} \theta_2(v, q^{1/2}) \tag{5.21}$$

Then, using the product representation

$$\theta_2(v, q^{1/2}) = q^{1/8}(z^{1/2} + z^{-1/2}) \prod_{n=1}^{\infty} (1 - q^n)(1 + q^n z)(1 + q^n z^{-1}) \tag{5.22}$$

we find

$$S_0 = \exp\left\{\sum_{n=1}^{\infty} \ln(1 - q^n)\right\} \frac{1}{2\pi i} \oint \frac{dz}{z} (z + 1) \times \exp\left\{\sum_{n=1}^{\infty} \{2 \ln(1 + zq^n) + \ln(1 + z^{-1} q^n)\}\right\} \tag{5.23}$$

We may now study the behavior of S_0 as $q \to 1$ by replacing the sums in (5.23) by integrals. Thus, using the definition $q = e^{-2\pi v/Mk_BT}$ and setting

$$x = \frac{2\pi v}{Mk_B T} \tag{5.24}$$

we find

$$\begin{aligned} S_0 \sim \exp &\left\{ \frac{Mk_B T}{2\pi v} \int_0^\infty dx \ln(1 - e^{-x}) \right\} \\ &\times \frac{1}{2\pi i} \oint \frac{dz}{z} (z+1) \\ &\times \exp \left\{ \frac{Mk_B T}{2\pi v} \int_0^\infty dx \, \{ 2 \ln(1 + ze^{-x}) + \ln(1 + z^{-1} e^{-x}) \} \right\} \end{aligned} \tag{5.25}$$

The integral over z may now be evaluated by steepest descents. The steepest descent point occurs at the values of z that satisfy

$$\ln(1+z)^2 = \ln(1 + z^{-1}) \tag{5.26}$$

and thus either $z = -1$ or

$$1 + z = z^{-1} \tag{5.27}$$

and hence we find that the steepest descents point is

$$z = \frac{\sqrt{5} - 1}{2} \tag{5.28}$$

Thus we have

$$\begin{aligned} S_0 \sim \exp &\left\{ \frac{Mk_B T}{2\pi v} \int_0^\infty dx \left\{ \ln(1 - e^{-x}) + 2 \ln\left(1 + \frac{\sqrt{5}-1}{2} e^{-x}\right) \right. \right. \\ &\left. \left. + \ln\left(1 + \frac{\sqrt{5}+1}{2} e^{-x}\right) \right\} \right\} \end{aligned} \tag{5.29}$$

which, if we define

$$t = e^{-x} \tag{5.30}$$

and recall the definition[46] of the dilogarithm

$$\mathrm{Li}_2(z) = -\int_0^z dt \, \frac{\ln(1-t)}{t} \tag{5.31}$$

may be rewritten as

$$S_0 \sim \exp\left\{-\frac{Mk_B T}{2\pi v}\left\{\mathrm{Li}_2(1)+2\,\mathrm{Li}_2\left(\frac{1-\sqrt{5}}{2}\right)+\mathrm{Li}_2\left(-\frac{\sqrt{5}+1}{2}\right)\right\}\right\} \tag{5.32}$$

Then if we note the special values of the dilogarithm

$$\mathrm{Li}_2(1)=\frac{\pi^2}{6} \tag{5.33a}$$

$$\mathrm{Li}_2\left(\frac{1-\sqrt{5}}{2}\right)=-\frac{\pi^2}{15}+\frac{1}{2}\ln^2\left(\frac{\sqrt{5}-1}{2}\right) \tag{5.33b}$$

$$\mathrm{Li}_2\left(-\frac{1+\sqrt{5}}{2}\right)=-\frac{\pi^2}{10}-\ln^2\left(\frac{\sqrt{5}+1}{2}\right) \tag{5.33c}$$

we obtain the result

$$S_0 \sim \exp\frac{Mk_B T\pi}{30v} \tag{5.34}$$

The identical behavior is obtained for S_1. Thus, with $Z=S_0(q)\,S_0(\bar{q})+S_1(q)\,S_1(\bar{q})$, we obtain from (5.5)–(5.9) an (effective) central charge of 2/5, which agrees with ref. 19.

APPENDIX A. GAUSSIAN CONSTRUCTION OF THE BRANCHING FUNCTIONS

The form of the branching function (2.27) may be simply obtained if we note that the Z_4 parafermions with the diagonal (A_5) modular invariant partition function is known to be the $r=(3/2)^{1/2}$ point on the orbifold line of $c=1$ conformal field theories (see, e.g., ref. 47). However, we are interested in the nondiagonal (D_4) partition function, which contains the current operator of dimension $(1, 0)$ in the spectrum. This model must therefore lie on the $c=1$ Gaussian line at the compactification radius $r=(3/2)^{1/2}$. Indeed, consider the general Gaussian model partition function

$$Z(r)=\frac{1}{\eta(q)\,\eta(\bar{q})}\sum_{m,n=-\infty}^{\infty} q^{\Delta_{m,n}}\bar{q}^{\bar{\Delta}_{m,n}} \tag{A.1}$$

where

$$\Delta_{m,n}(r)=\frac{1}{2}\left(\frac{m}{2r}+nr\right)^2 \quad \text{and} \quad \bar{\Delta}_{m,n}(r)=\frac{1}{2}\left(\frac{m}{2r}-nr\right)^2 \tag{A.2}$$

and set $r=(3/2)^{1/2}$ to obtain

$$Z\left(\left(\frac{3}{2}\right)^{1/2}\right)=\frac{1}{\eta(q)\,\eta(\bar{q})}\sum_{m,n=-\infty}^{\infty} q^{3((m+3n)/6)^2}\bar{q}^{3((m-3n)/6)^2} \tag{A.3}$$

Rewriting the sum as

$$Z\left(\left(\frac{3}{2}\right)^{1/2}\right)=\frac{1}{\eta(q)\,\eta(\bar{q})}\sum_{b=0}^{5}\sum_{\substack{m,n=-\infty\\ m+n\equiv b \bmod 6}}^{\infty} q^{3((m+n)/6)^2}\bar{q}^{3(m-n)/6)^2} \tag{A.4}$$

we see that

$$Z\left(\left(\frac{3}{2}\right)^{1/2}\right)=\frac{1}{\eta(q)\,\eta(\bar{q})}\sum_{b=0}^{5} f_{3,b}(q)\, f_{3,b}(\bar{q}) \tag{A.5}$$

with $f_{m,n}$ defined by (2.29). Then, nothing the symmetry

$$f_{3,1}=f_{3,5} \qquad \text{and} \qquad f_{3,2}=f_{3,4} \tag{A.6}$$

we obtain precisely the Z_{pf4} of (1.10) with the expressions of (2.27) for $b_0^0+b_4^0$, b_2^0, b_0^2, and b_2^2.

APPENDIX B. BRANCHING FUNCTIONS OF $(A_3^{(1)})_1 \times (A_3^{(1)})_1/(A_3^{(1)})_2$

The branching functions (2.27) can also be expressed as a three-dimensional sum in the following way.[30–32] Let α_i, $i=1, 2, 3$, be the simple roots of A_3. Then

$$(\alpha_i, \alpha_j)=c_{i,j} \tag{B.1}$$

where c is the Cartan matrix of A_3

$$c=\begin{pmatrix} 2 & -1 & 0\\ -1 & 2 & -1\\ 0 & -1 & 2\end{pmatrix} \tag{B.2}$$

Let λ_i be the fundamental weights of A_3, so that $\lambda_j=\sum_k c_{j,k}^{-1}\alpha_k$. The dominant weights $\mathbf{a}^{(k)}$ of level k are defined to be

$$\mathbf{a}^{(k)}=\sum_{i=1}^{3} a_i\lambda_i, \qquad a_i\in \mathbf{Z}, \qquad a_i\geqslant 0, \qquad \sum_{i=1}^{3} a_i\leqslant k \tag{B.3}$$

Let $\mathbf{r} = \mathbf{a}^{(1)} + \rho$ and $\mathbf{s} = \mathbf{a}^{(2)} + \rho$, where $\rho = \sum_i \lambda_i$. Then the following branching functions are identical with (2.27):

$$b_{\mathbf{r},\mathbf{s}} = \frac{1}{\eta^3} \sum_{\mathbf{k} \in \mathbf{Q}} \sum_{w \in W} \det(w)\, q^{|30\mathbf{k} - 6\mathbf{r} + 5w(\mathbf{s})|^2/60} \tag{B.4}$$

where W is the Weyl group of A_3, generated by the three simple reflections

$$\sigma_i(\beta) = \beta - (\alpha_i, \beta)\alpha_i, \qquad i = 1, 2, 3 \tag{B.5}$$

This group has 24 elements made up of powers of the simple reflections, and $\det(w) = \pm 1$, depending on whether the minimum number of simple reflections making up the element w is even or odd. $\mathbf{Q}$ is the root lattice of A_3, i.e., $\mathbf{k} = \sum_i m_i \alpha_i$, and the sum $\sum_{\mathbf{k} \in \mathbf{Q}}$ is thus a sum over m_i, $i = 1, 2, 3$, from $-\infty$ to ∞.

Equation (B.4) gives seven unique branching functions, but only five appear in our model: these can be obtained by setting $\mathbf{r} = \rho$ and choosing the following $\mathbf{a}^{(2)} = \mathbf{s} - \rho$:

$$\mathbf{a}^{(2)} = \begin{cases} 0, & b_{\mathbf{r},\mathbf{s}} = b_0^0, \quad h_{\mathbf{r},\mathbf{s}} = 0 \\ 2\lambda_2, & b_{\mathbf{r},\mathbf{s}} = b_4^0, \quad h_{\mathbf{r},\mathbf{s}} = 1 \\ 2\lambda_1, & b_{\mathbf{r},\mathbf{s}} = b_2^0, \quad h_{\mathbf{r},\mathbf{s}} = 3/4 \\ \lambda_2 & b_{\mathbf{r},\mathbf{s}} = b_2^2, \quad h_{\mathbf{r},\mathbf{s}} = 1/12 \\ \lambda_1 + \lambda_3, & b_{\mathbf{r},\mathbf{s}} = b_0^2, \quad h_{\mathbf{r},\mathbf{s}} = 1/3 \end{cases} \tag{B.6}$$

Here, the conformal dimension $h_{\mathbf{r},\mathbf{s}}$ is defined to be

$$h_{\mathbf{r},\mathbf{s}} = -\frac{1}{12} + \frac{|-6\mathbf{r} + 5\mathbf{s}|^2}{60} \tag{B.7}$$

The three-dimensional sum (B.4) is to be compared with the expressions (3.13) and (4.16) of the text. We note that the sum (B.4) has a power $q^{\sum 30 m_j^2}$ and thus the powers of q grow much more rapidly as a function of m_j than do (3.13) and (4.16). Note also that when the sum over W is performed, (B.4) contains 24 triple sums, whereas (3.13) has only one and (4.16) has two.

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REFERENCES

1. H. Bethe, *Z. Phys.* **71**:205 (1931).
2. A. A. Belavin, A. M. Polyakov, and A. B. Zamolodchikov, *J. Stat. Phys.* **34**:763 (1984); *Nucl. Phys. B* **241**:333 (1984).
3. J. L. Cardy, *Nucl. Phys. B* **270**:186 (1986).
4. A. Cappelli, C. Itzykson, and J.-B. Zuber, *Nucl. Phys. B* **280**:445 (1987).
5. B. L. Feigin and D. B. Fuchs, *Funct. Anal. Appl.* **17**:241 (1983).
6. A. Rocha-Caridi, In *Vertex Operators in Mathematics and Physics*, J. Lepowsky, S. Mandelstam, and I. Singer, eds. (Springer, Berlin, 1985), p. 451.
7. V. G. Kac and D. H. Peterson, *Adv. Math.* **53**:125 (1984).
8. V. G. Kac and M. Wakimoto, *Adv. Math.* **70**:156 (1988).
9. A. Klümper and P. A. Pearce, *J. Stat. Phys.* **64**:13 (1991).
10. A. Klümper and P. A. Pearce, *Physica A* **183**:304 (1992).
11. V. Pasquier, *Nucl. Phys. B* **285**[FS19]:162 (1987).
12. P. A. Pearce, private communication.
13. G. Albertini, S. Dasmahapatra, and B. M. McCoy, *Phys. Lett. A* **170**: 397 (1992).
14. D. Gepner and Z. Qiu, *Nucl. Phys.* **B285**[FS19]:423 (1987).
15. J. Lepowsky and M. Primc, *Structure of the Standard Modules for the Affine Lie Algebra* $A_1^{(1)}$ (AMS, Providence, Rhode Island, 1985).
16. G. Albertini, *J. Phys. A* **25**:1799 (1992).
17. G. Albertini, S. Dasmahapatra, and B. M. McCoy, *Int. J. Mod. Phys. A* **7** Suppl. 1A):1 (1992).
18. J. L. Cardy, *Phys. Rev. Lett.* **54**:1354 (1985).
19. C. Itzykson, H. Saleur, and J.-B. Zuber, *Europhys. Lett.* **2**:91 (1986).
20. T. D. Lee and C. N. Yang, *Phys. Rev.* **87**:410 (1952).
21. G. Albertini, B. M. McCoy, and J. H. H. Perk, *Phys. Lett. A* **135**:159 (1989).
22. R. J. Baxter, V. V. Bazhanov, and J. H. H. Perk, *Int. J. Mod. Phys. B* **4**:803 (1990).
23. P. A. Pearce, *Int. J. Mod. Phys. A* **7** (Suppl. 1B):791 (1992).
24. P. Fendley and P. Ginsparg, *Nucl. Phys. B* **324**:549 (1989).
25. Ph. Roche, *Commun. Math. Phys.* **127**:395 (1990).
26. J. Lepowsky and R. L. Wilson, *Proc. Natl. Acad. Sci. USA* **78**:7254 (1981).
27. A. B. Zamolodchikov and V. A. Fateev, *Sov. Phys. JETP* **62**:215 (1985).
28. J. Distler and Z. Qiu, *Nucl. Phys. B* **336**:533 (1990).
29. V. A. Fateev and S. L. Lykyanov, *Int. J. Mod. Phys. A* **3**:507 (1988).
30. M. Jimbo, T. Miwa, and M. Okado, *Nucl. Phys. B* **300**[FS22]:74 (1988).
31. P. Christe and F. Ravanini, *Int. J. Mod. Phys. A* **4**:897 (1989).
32. V. A. Fateev and S. L. Lykyanov, *Sov. Sci. Rev. A Phys.* **15**:1 (1990).
33. D. Altschuler, M. Bauer, and H. Saleur, *J. Phys. A* **23**:L789 (1990).
34. R. Kedem, T. R. Klassen, B. M. McCoy, and E. Melzer, *Phys. Lett. B*, in press.

35. F. D. M. Haldane, *Phys. Rev. Lett.* **67**:937 (1991).
36. H. W. J. Blöte, J. L. Cardy, and M. P. Nightingale, *Phys. Rev. Lett.* **56**:742 (1986).
37. I. Affleck, *Phys. Rev. Lett.* **56**:746 (1986).
38. B. Richmond and G. Szekeres, *J. Aust. Math. Soc. Ser. A* **31**:362 (1981).
39. B. M. McCoy and S.-S. Roan, *Phys. Lett. A* **150**:347 (1990).
40. S. Howes, L. P. Kadanoff, and M. den Nijs, *Nucl. Phys. B* **215**[FS7]:169 (1983).
41. G. Albertini, B. M. McCoy, and J. H. H. Perk, In *Advanced Studies in Pure Mathematics*, Vol. 19 (Kinokuniya/Academic Press, Tokyo, 1989), p. 1.
42. L. J. Rogers, *Proc. Lond. Math. Soc.* (Series 1) **25**:318 (1894) and (Series 2) **16**:315 (1917).
43. I. Schur, *Berliner Sitzungberichte* **23**:301 (1917).
44. L. J. Rogers, S. Ramanujan and G. H. Hardy, *Proc. Camb. Phil. Soc.* **19**:211 (1919).
45. A. Erdélyi, *Higher Transcendental Functions*, Vol. 2 (McGraw-Hill, New York, 1953), p. 355.
46. L. Lewin, *Dilogarithms and Associated Functions* (MacDonald, London, 1958).
47. P. Ginsparg, In *Fields, Strings and Critical Phenomena, Les Houches 1988*, E. Brèzin and J. Zinn-Justin, eds. (North-Holland, Amsterdam, 1989).